

On common quadratic Lyapunov functions for stable discrete-time LTI systems

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Abstract

This paper deals with the question of the existence of weak and strong common quadratic Lyapunov functions (CQLFs) for stable discrete-time linear time-invariant (LTI) systems. The main result of the paper provides a simple characterisation of pairs of such systems for which a weak CQLF of a given form exists but for which no strong CQLF exists. An application of this result to second order discrete-time LTI systems is presented.

1 Introduction and Notation

In recent years, the area of systems theory and control has witnessed a considerable growth of interest in systems characterised by a combination of continuous dynamics and logic-based switching. Systems of this type are commonly referred to as switching systems in the scientific and engineering literature. A major issue in this area is the determination of easily verifiable and interpretable conditions that guarantee the stability of such systems. For

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an overview of some of the more recent approaches to this issue see (DeCarlo et al. 2000), (Branicky 1994), (Ye, Michel and Hou 1998), (Shorten and Narendra 2000), (Shorten and Narendra 1998), (Johansson and Rantzer 1998), (Shorten and O’Cairbre 2002), (O’Cairbre and Shorten 2001), (Narendra and Balakrishnan 1994b). In this context the problem of the existence or non-existence of a common quadratic Lyapunov function (CQLF) for a family of linear time-invariant (LTI) systems is of great importance. There is already a considerable body of literature dedicated to this question for both discrete-time and continuous-time systems (Boyd and Yang 1989), (Ando 2001), (Agrachev and Liberzon 2001), (Shorten and Narendra 1999), (Akar and Narendra 2001), (Gurvits 1995), (Ooba and Funahashi 1999), (Ooba and Funahashi 1997), (Mori, Mori and Kuroe 1998), (Mori, Mori and Kuroe 2001), (Narendra and Balakrishnan 1994a). The main result presented here is concerned with the CQLF existence problem for a family of two discrete-time LTI systems.

Throughout \mathbb{R} and \mathbb{C} will denote the fields of real and complex numbers respectively and $\mathbb{R}^{n \times n}$ ($\mathbb{C}^{n \times n}$) denotes the space of $n \times n$ matrices with real (complex) entries. For a matrix A in $\mathbb{R}^{n \times n}$, A^T denotes its transpose, $\det(A)$ its determinant and a_{ij} the entry in the (i, j) position of A . Similarly for a vector x in \mathbb{R}^n , x_i denotes the i^{th} component of x . A matrix $A \in \mathbb{R}^{n \times n}$ is said to be symmetric if $A = A^T$. The notation $P > 0$ ($P \geq 0$) is used to denote that the matrix P is positive (semi-)definite with $P < 0$, ($P \leq 0$) meaning that $-P > 0$ ($-P \geq 0$).

We shall be concerned with the discrete-time Lyapunov matrix equation ¹

$$A^T P A - P = -Q \quad A, P, Q \in \mathbb{R}^{n \times n} \tag{1}$$

The next result is standard in the theory of discrete-time LTI systems (Rugh 1996).

¹referred to as the Stein equation by some authors

Theorem 1.1 : Let $A \in \mathbb{R}^{n \times n}$. If there exist $P = P^T > 0$, $Q = Q^T > 0$ satisfying (1), then all of the eigenvalues of the matrix A have magnitude less than 1. Conversely, if all of the eigenvalues of A have magnitude less than 1, then there exists a Hermitian solution P to (1) for any choice of Hermitian Q and moreover, if $Q > 0$ then $P > 0$.

The relevance of equation (1) to the theory of discrete-time LTI systems (and the reason for calling it the discrete-time Lyapunov equation) is provided by the fact (Rugh 1996), (Kailath 1980) that the discrete-time LTI system

$$\Sigma_A : x(k+1) = Ax(k) \quad x(0) = x_0, \quad A \in \mathbb{R}^{n \times n} \quad (2)$$

is exponentially stable if there exist positive definite matrices P, Q satisfying (1). We shall refer to such systems as discrete-time stable systems from now on. Note that a system Σ_A is discrete-time stable if and only if all eigenvalues of the matrix A have magnitude less than 1.

In the spirit of (Shorten, Narendra and Mason 2003), we now define strong and weak CQLFs for a set of discrete-time stable LTI systems.

Strong and weak common quadratic Lyapunov functions

Consider the set of discrete-time stable LTI systems

$$\Sigma_{A_i} : x(k+1) = A_i x(k), \quad i \in \{1, 2, \dots, M\}. \quad (3)$$

where the A_i , $i \in \{1, 2, \dots, M\}$, are constant matrices in $\mathbb{R}^{n \times n}$ such that all the eigenvalues of A_i have magnitude less than 1. Let the matrix $P = P^T > 0$, $P \in \mathbb{R}^{n \times n}$, be a simultaneous solution to the discrete-time Lyapunov equations

$$A_i^T P A_i - P = -Q_i, \quad i \in \{1, 2, \dots, M\}. \quad (4)$$

If Q_i is positive definite, the function $V(x) = x^T P x$ is a *strong quadratic Lyapunov function* for the system Σ_{A_i} . If Q_i is positive definite for all i , $1 \leq i \leq n$, $V(x)$ is a *strong CQLF* for the family of systems Σ_{A_i} , $1 \leq i \leq n$.

Similarly, if Q_i is positive semi-definite and singular the function $V(x) = x^T P x$ is a *weak quadratic Lyapunov function* for the system Σ_{A_i} . If Q_i is positive semi-definite and singular for all $i, 1 \leq i \leq n$, $V(x)$ is a *weak CQLF* for the family of systems $\Sigma_{A_i}, 1 \leq i \leq n$.

The notion of a matrix pencil will be convenient for expressing our later results, so we now recall the definition of this concept.

The matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$

The matrix pencil $\sigma_{\gamma[0,\infty)}[A_1, A_2]$ is the parameterised family of matrices

$$\sigma_{\gamma[0,\infty)}[A_1, A_2] = \{A_1 + \gamma A_2, \gamma \in [0, \infty)\}, \quad (5)$$

where $A_1, A_2 \in \mathbb{R}^{n \times n}$. We say that the pencil is non-singular if $A_1 + \gamma A_2$ is non-singular for all $\gamma \geq 0$. Otherwise the pencil is said to be singular.

To complete this section, we recall some facts about the continuous-time Lyapunov equation that will be needed later. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be *Hurwitz* if all of its eigenvalues lie in the open left half of the complex plane. The following standard result is the continuous-time analogue of Theorem 1.1 (Horn and Johnson 1991), (Rugh 1996).

Theorem 1.2 : Let $A \in \mathbb{R}^{n \times n}$. If there exist positive definite matrices P and Q satisfying the continuous-time Lyapunov equation

$$A^T P + P A = -Q \quad (6)$$

then A is Hurwitz. Conversely, if A is Hurwitz, then there exists a Hermitian solution P to the equation (6) for any choice of Hermitian matrix Q and moreover, if $Q > 0$ then $P > 0$.

As in the discrete-time case, equation (6) is intimately related to the stability properties of the continuous-time LTI system

$$\Sigma_A : \frac{dx}{dt} = Ax \quad x(0) = x_0, \quad A \in \mathbb{R}^{n \times n}.$$

Using equation (6), the notions of strong and weak CQLFs are defined for continuous-time LTI systems in the obvious way.

2 Some Preliminary Lemmas

In this section, we present some technical results that shall prove useful in establishing the main theorem of the next section. While a number of these results are known, we include them here for the sake of completeness. The following well-known Lemma provides a convenient test for singularity of a matrix pencil.

Lemma 2.1 : Let A_1, A_2 be non-singular matrices in $\mathbb{R}^{n \times n}$. The pencil $\sigma_{\gamma[0, \infty)}[A_1, A_2]$ is singular if and only if the matrix product $A_1 A_2^{-1}$ has a negative (real) eigenvalue.

Proof: A_1 is non-singular so the pencil is not singular for $\gamma = 0$. Now if $\gamma > 0$ then

$$\begin{aligned} \det(A_1 + \gamma A_2) = 0 &\iff (\det(A_1 A_2^{-1} + \gamma I))(\det A_2) = 0 \\ &\iff \det(A_1 A_2^{-1} + \gamma I) = 0 \end{aligned}$$

Thus the matrix $A_1 + \gamma A_2$ is singular if and only if the matrix product $A_1 A_2^{-1}$ has the negative (real) eigenvalue $-\gamma$. \square

We next record the simple observation that the quadratic Lyapunov functions for the discrete-time stable LTI systems Σ_A and Σ_{-A} coincide. This relates to the result in (Loewy 1976) identifying the quadratic Lyapunov functions for Σ_A and $\Sigma_{A^{-1}}$ for continuous-time systems.² A related observation was also made in (Barker, Berman and Plemmons 1978).

Lemma 2.2 : Consider the discrete-time stable LTI systems

$$\begin{aligned} \Sigma_A : x(k+1) &= Ax(k) \\ \Sigma_{-A} : x(k+1) &= (-A)x(k). \end{aligned}$$

Then, any quadratic Lyapunov function for Σ_A is also a quadratic Lyapunov function for Σ_{-A} .

²In fact, it is shown in (Loewy 1976) that, up to constant factors, A and A^{-1} are the only matrices in $\mathbb{R}^{n \times n}$ sharing the same set of real solutions to the continuous time Lyapunov equation (6)

Proof: This is a straightforward consequence of the fact that

$$A^T P A - P = (-A)^T P (-A) - P$$

for any positive definite matrix P . \square

The following lemma relates the discrete-time Lyapunov equation (1) to the continuous-time Lyapunov equation (6), and introduces a notation that we shall use to state our results.

Lemma 2.3 : Let Σ_A be a discrete-time stable LTI system, and $P = P^T > 0$ be a solution of the discrete-time Lyapunov equation

$$A^T P A - P = -Q \quad Q > 0.$$

Consider the matrix

$$C(A) = (A - I)(A + I)^{-1}$$

(Note that this is well defined as the eigenvalues of A are all of modulus less than 1.) Then P is also a solution of the continuous-time Lyapunov equation

$$C(A)^T P + P C(A) = -Q'$$

with $Q' = 2(A + I)^{-T} Q (A + I)^{-1} > 0$ The mapping

$$A \rightarrow C(A) = (A - I)(A + I)^{-1}$$

is known as the *bilinear transformation* in the engineering literature ³ (Kailath 1980), (Gajic and Qureshi 1995) and it provides a link between the discrete-time and the continuous-time Lyapunov equations. Henceforth, we shall use the notation $C(A)$ for this mapping and write $C^{-1}(A)$ for the inverse mapping $C^{-1}(A) = (I + A)(I - A)^{-1}$.

The next result gives a simple necessary condition for the existence of a strong CQLF for two discrete-time stable LTI systems.

³It is more commonly referred to as the Cayley transform in mathematics

Lemma 2.4 : Let $\Sigma_{A_1}, \Sigma_{A_2}$ be two discrete-time stable LTI systems with a strong CQLF given by $V(x) = x^T P x$. Then the two matrix pencils $\sigma_{\gamma[0, \infty)}[C(A_1), C(A_2)], \sigma_{\gamma[0, \infty)}[C(A_1), C(A_2)^{-1}]$ are non-singular.

Proof: It is straightforward to check that $C(-A) = C(A)^{-1}$. Thus, by Lemma 2.2 and Lemma 2.3, P also defines a strong CQLF for the continuous-time systems $\Sigma_{C(A_1)}, \Sigma_{C(A_2)}$ and for the continuous-time systems $\Sigma_{C(A_1)}, \Sigma_{C(A_2)^{-1}}$. This implies in turn that P defines a strong quadratic Lyapunov function for the systems $\Sigma_{C(A_1) + \gamma C(A_2)}, \Sigma_{C(A_1) + \gamma C(A_2)^{-1}}$ for all $\gamma > 0$. From this it follows that the pencils $\sigma_{\gamma[0, \infty)}[C(A_1), C(A_2)], \sigma_{\gamma[0, \infty)}[C(A_1), C(A_2)^{-1}]$ must consist entirely of Hurwitz matrices (Theorem 1.2) and, in particular must be non-singular. \square

The proofs of the next two results to be presented are a little longer than the others in this section, and for this reason are included in the appendix rather than in the body of the text.

Lemma 2.5 : Let $u, v, x, y \in \mathbb{R}^n$ be any four non-zero vectors. There exists a non-singular $T \in \mathbb{R}^{n \times n}$ such that each component of the vectors Tu, Tv, Tx, Ty is non-zero.

The next result (Lemma 2.6) establishes a convenient relationship between two parameterizations of the same hyperplane in the space of symmetric matrices in $\mathbb{R}^{n \times n}$. A direct proof of this result is provided in the appendix.

Lemma 2.6 : Let x, y, u, v be 4 non-zero vectors in \mathbb{R}^n such that for all $n \times n$ real symmetric matrices H

$$x^T H y = -k u^T H v$$

with $k > 0$. Then either

$$x = \alpha u \text{ for some real scalar } \alpha, \text{ and } y = -\left(\frac{k}{\alpha}\right)v$$

or

$$x = \beta v \text{ for some real scalar } \beta \text{ and } y = -\left(\frac{k}{\beta}\right)u.$$

3 Main results

The principal result of this paper concerns two discrete-time stable LTI systems for which no strong CQLF exists but for which a weak CQLF exists with each of the $Q_i, i \in \{1, 2\}$ in (4) of rank $n - 1$. In Theorem 3.1 we provide a simple algebraic characterisation of this situation. The result is of interest for any class of systems where the transition from the existence of a CQLF to the non-existence of a CQLF passes through the situation described in the theorem.

Remark: It is possible to show that for any matrix $A \in \mathbb{R}^{n \times n}$, all of whose eigenvalues are of modulus less than one, the set of matrices $P = P^T$ satisfying

$$A^T P A - P = -Q \quad Q \geq 0, \text{rank}(Q) = n - 1$$

is dense in the set of matrices satisfying

$$A^T P A - A = -Q \quad Q \geq 0, \det(Q) = 0.$$

This indicates that the situation described in the theorem is potentially of great importance in providing insight into the existence question for strong and weak CQLFs.

Theorem 3.1 : Let $\Sigma_{A_1}, \Sigma_{A_2}$ be two discrete-time stable LTI systems such that a positive semi-definite solution $P = P^T \geq 0$ exists to the non-strict Lyapunov equations

$$A_1^T P A_1 - P = -Q_1 \leq 0, \tag{7}$$

$$A_2^T P A_2 - P = -Q_2 \leq 0, \tag{8}$$

for some positive semi-definite matrices Q_1, Q_2 both of rank $n - 1$ ($n \geq 2$). Furthermore suppose that the systems $\Sigma_{A_1}, \Sigma_{A_2}$ do not have a strong CQLF. Under these conditions, at least one of the pencils $\sigma_{\gamma[0, \infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0, \infty)}[C(A_1), C(A_2)^{-1}]$ is singular, and equivalently, at least one of the matrix products $C(A_1)C(A_2)$ and $C(A_1)C(A_2)^{-1}$ has a real negative eigenvalue.

Proof: As Q_1 and Q_2 are of rank $n - 1$, there are non-zero vectors x_1, x_2 such that

$$x_1^T Q_1 x_1 = 0 \quad (9)$$

$$x_2^T Q_2 x_2 = 0. \quad (10)$$

The proof of Theorem 3.1 is split into two main stages.

Stage 1 : The first stage in the proof is to show that if there exists a real symmetric matrix \bar{P} satisfying

$$x_1^T (A_1^T \bar{P} A_1 - \bar{P}) x_1 < 0 \quad (11)$$

$$x_2^T (A_2^T \bar{P} A_2 - \bar{P}) x_2 < 0 \quad (12)$$

then $\Sigma_{A_1}, \Sigma_{A_2}$ would have a strong CQLF.

So assume that there is some \bar{P} satisfying (11), (12), and consider the set

$$\Omega_1 = \{x \in \mathbb{R}^n : \|x\| = 1 \text{ and } x^T (A_1^T \bar{P} A_1 - \bar{P}) x \geq 0\}.$$

(Here $\|x\|$ is the usual Euclidean norm on \mathbb{R}^n .)

We shall show that there is a positive constant $C_1 > 0$ such that $A_1^T (P + \delta_1 \bar{P}) A_1 - (P + \delta_1 \bar{P})$ is negative definite provided that $0 < \delta_1 < C_1$.

Firstly suppose that Ω_1 was empty. Then $A_1^T (P + \delta_1 \bar{P}) A_1 - (P + \delta_1 \bar{P})$ is negative definite for any $\delta_1 > 0$. So any positive constant C_1 will work in this case.

Now, assume that the set Ω_1 is non-empty. The function that takes x to $x^T (A_1^T \bar{P} A_1 - \bar{P}) x$ is continuous. Thus Ω_1 is closed and bounded, hence compact. Furthermore x_1 (or any non-zero multiple of x_1) is not in Ω_1 and thus $x^T (A_1^T \bar{P} A_1 - \bar{P}) x$ is strictly negative on Ω_1 .

Let M_1 be the maximum value of $x^T (A_1^T \bar{P} A_1 - \bar{P}) x$ on Ω_1 , and let M_2 be the maximum value of $x^T (A_1^T P A_1 - P) x$ on Ω_1 . Then by the final remark in the previous paragraph, $M_2 < 0$.

Choose any constant $\delta_1 > 0$ such that

$$\delta_1 < \frac{|M_2|}{M_1 + 1}$$

and consider the matrix

$$P + \delta_1 \bar{P}.$$

By separately considering the cases $x \in \Omega_1$ and $x \notin \Omega_1$, $\|x\| = 1$, it is easy to see that for all non-zero vectors x of norm 1

$$x^T (A_1^T (P + \delta_1 \bar{P}) A_1 - (P + \delta_1 \bar{P})) x < 0$$

provided $0 < \delta_1 < \frac{|M_2|}{M_1 + 1}$. Let C_1 denote the value $\frac{|M_2|}{M_1 + 1}$. Thus we have shown that there is some positive constant C_1 such that $A_1^T (P + \delta_1 \bar{P}) A_1 - (P + \delta_1 \bar{P})$ is negative definite provided that $0 < \delta_1 < C_1$.

Now the same argument can be used to guarantee the existence of a positive constant C_2 such that

$$x^T (A_2^T (P + \delta_1 \bar{P}) A_2 - (P + \delta_1 \bar{P})) x < 0.$$

for all non-zero x provided we choose $0 < \delta_1 < C_2$. So, if we choose δ less than the minimum of C_1, C_2 , we would have a real symmetric matrix

$$P_1 = P + \delta \bar{P}$$

satisfying (4) with $Q_1, Q_2 > 0$. Furthermore it follows from Theorem 1.1 that $P_1 > 0$ and thus $V(x) = x^T P_1 x$ would be a strong CQLF for $\Sigma_{A_1}, \Sigma_{A_2}$.

Stage 2 : So, under our assumptions there is no real symmetric matrix H such that

$$x_1^T (A_1^T H A_1 - H) x_1 < 0 \tag{13}$$

$$x_2^T (A_2^T H A_2 - H) x_2 < 0. \tag{14}$$

This means that the mapping from the space of real symmetric matrices in $\mathbb{R}^{n \times n}$ into \mathbb{R}^2 defined by

$$H \rightarrow \begin{pmatrix} x_1^T (A_1^T H A_1 - H) x_1 \\ x_2^T (A_2^T H A_2 - H) x_2 \end{pmatrix}$$

must be of rank 1.

Therefore there is some positive constant k such that

$$x_1^T (A_1^T H A_1 - H) x_1 = -k x_2^T (A_2^T H A_2 - H) x_2 \quad (15)$$

for all real symmetric matrices H .

Expanding the expression $(A_1 x_1 - x_1)^T H (A_1 x_1 + x_1)$ for symmetric H gives

$$\begin{aligned} x_1^T A_1^T H A_1 x_1 - x_1^T H x_1 + x_1^T A_1^T H x_1 - x_1^T H A_1 x_1 &= x_1^T A_1^T H A_1 x_1 - x_1^T H x_1 \\ &= x_1^T (A_1^T H A_1 - H) x_1 \end{aligned}$$

Therefore

$$x_1^T (A_1^T H A_1 - H) x_1 = (A_1 x_1 - x_1)^T H (A_1 x_1 + x_1) \quad (16)$$

and similarly

$$x_2^T (A_2^T H A_2 - H) x_2 = (A_2 x_2 - x_2)^T H (A_2 x_2 + x_2) \quad (17)$$

for all symmetric H .

Combining (15) with (16) and (17) yields the following identity for all symmetric matrices H .

$$(A_1 x_1 - x_1)^T H (A_1 x_1 + x_1) = -k (A_2 x_2 - x_2)^T H (A_2 x_2 + x_2) \quad (18)$$

Now apply Lemma 2.6 to deduce that either

$$(A_1 x_1 + x_1) = \alpha (A_2 x_2 + x_2), \quad (A_1 x_1 - x_1) = -\frac{k}{\alpha} (A_2 x_2 - x_2) \quad (19)$$

or

$$(A_1x_1 + x_1) = \alpha(A_2x_2 - x_2), \quad (A_1x_1 - x_1) = -\frac{k}{\alpha}(A_2x_2 + x_2) \quad (20)$$

In the first case (19), we have

$$x_1 = \alpha(A_1 + I)^{-1}(A_2 + I)x_2$$

and substituting this into the second identity in (19) yields

$$(A_1 - I)(A_1 + I)^{-1}(A_2 + I)x_2 = -\frac{k}{\alpha^2}(A_2 - I)x_2$$

Letting $y = (A_2 + I)x_2$ we see that

$$((A_1 - I)(A_1 + I)^{-1} + \frac{k}{\alpha^2}(A_2 - I)(A_2 + I)^{-1})y = 0$$

and hence the pencil $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ is singular and the product $C(A_1)C(A_2)^{-1}$ has a negative eigenvalue. A similar argument shows that in the case (20), the pencil $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ is singular and the product $C(A_1)C(A_2)$ has a negative eigenvalue. This completes the proof of Theorem 3.1. \square

We note that an alternative proof of Theorem 3.1 is possible by combining the bilinear transformation with results presented in (Shorten, Narendra and Mason 2003).

Comments on the Applicability of the result:

Theorem 3.1 provides an insight into the general question of CQLF existence for pairs of discrete-time LTI systems and can be used to derive checkable necessary and sufficient conditions for CQLF existence for specific classes of systems of practical importance. The general approach to applying the result relies on the following two facts.

- (i) If two systems $\Sigma_{A_1}, \Sigma_{A_2}$ have a strong CQLF, then the matrix pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ are Hurwitz.
- (ii) For two systems which do not have a strong CQLF, there is some $d > 0$ such that $\Sigma_{A_1}, \Sigma_{C^{-1}(C(A_1)-dI)}$ have a weak CQLF but no strong CQLF.

In order to apply Theorem 3.1, it is necessary to identify systems classes such that the systems obtained in point (ii) above will satisfy the hypotheses of the theorem. To illustrate the method, one such class is presented in the next section. It is the authors belief that other classes exist and the identification of such classes is currently the subject of ongoing research.

It is important to note that for classes to which the result applies in the above sense, it would be possible to derive conditions in terms of the eigenvalue locus of simple matrix pencils which were equivalent to the existence of a strong CQLF. Such conditions are easily checkable and furthermore have a strong connection with the stability properties of the associated switching system. Indeed, the results presented in (Shorten, O’Cairbre and Curran 2000) indicate how such conditions can inform us as to whether or not CQLF existence is a conservative condition for stability. This is of considerable practical importance in using CQLFs to test stability for switching systems.

4 Second order systems

In this section we present an example to illustrate the use of Theorem 3.1.

Example: Second order systems

Let Σ_{A_1} and Σ_{A_2} be discrete-time stable LTI systems with $A_1, A_2 \in \mathbb{R}^{2 \times 2}$. We note the following readily verifiable facts.

- (a) If a CQLF exists for Σ_{A_1} and Σ_{A_2} then the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ and $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ consist entirely of Hurwitz matrices.
- (b) If a CQLF does not exist for Σ_{A_1} and Σ_{A_2} then a CQLF (in the continuous-time sense) does not exist for the continuous-time systems $\Sigma_{C(A_1)}, \Sigma_{C(A_2)}$. However by choosing $d > 0$ sufficiently large, we can ensure that a CQLF (in the continuous-time sense) exists for $\Sigma_{C(A_1)-dI}$ and $\Sigma_{C(A_2)}$. A continuity argument can be employed to show that there is

some d_1 with $0 < d_1 < d$ such that $C^{-1}(C(A_1) - d_1 I)$ and A_2 satisfy Theorem 3.1, and thus one of the pencils $\sigma_{\gamma[0,\infty)}[C(A_1) - d_1 I, C(A_2)]$ and $\sigma_{\gamma[0,\infty)}[C(A_1) - d_1 I, C(A_2)^{-1}]$ is necessarily singular. Hence, it follows that one of the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ is not Hurwitz.

Items (a) and (b) establish the following facts. Given two discrete-time stable second order LTI systems Σ_{A_1} and Σ_{A_2} , a necessary condition for the existence of a CQLF is that the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ and $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ are Hurwitz. Conversely, if a CQLF does not exist for Σ_{A_1} , Σ_{A_2} , then one of the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$, $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ is not Hurwitz. Together these conditions yield the following result which is closely related to that presented in (Akar and Narendra 2001) and is the discrete-time counterpart of results presented in (Shorten and Narendra 1999).

A necessary and sufficient condition for the discrete-time stable LTI systems Σ_{A_1} and Σ_{A_2} , $A_1, A_2 \in \mathbb{R}^{2 \times 2}$, to have a CQLF is that the pencils $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)]$ and $\sigma_{\gamma[0,\infty)}[C(A_1), C(A_2)^{-1}]$ are Hurwitz.

An example of a real world application where the above result can be applied is the problem of automatic gearbox control. For more information on this topic see the thesis (Shorten 1996).

5 Concluding remarks

In this note we have derived a CQLF non-existence theorem. We have applied this theorem to derive a CQLF existence result for a pair of stable LTI systems that belong to a certain system class. We believe that our result can be applied to derive similar results for pairs of stable LTI systems belonging to other important system classes.

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Appendix

Proof of Lemma 2.5

Consider the norm $\|A\|_\infty = \sup\{|a_{ij}| : 1 \leq i, j \leq n\}$ on $\mathbb{R}^{n \times n}$, and let z be any non-zero vector in \mathbb{R}^n . Then it is easy to see that the set $\{T \in \mathbb{R}^{n \times n} : \det(T) \neq 0, (Tz)_i \neq 0, 1 \leq i \leq n\}$ is open. On the other hand, if $T \in \mathbb{R}^{n \times n}$ is such that $(Tz)_i = 0$ for some i , an arbitrarily small change in an appropriate element of the i^{th} row of T will result in a matrix T' such that $(T'z)_i \neq 0$. From this it follows that arbitrarily close to the original matrix T , there is some $T_1 \in \mathbb{R}^{n \times n}$ such that $T_1 z$ is non-zero component-wise.

Now to prove the lemma, simply select a non-singular T_0 such that $T_0 x$ is non-zero component-wise. Suppose that some component of $T_0 y$ is zero. By the arguments in the previous paragraph, it is clear that we can select a non-singular $T_1 \in \mathbb{R}^{n \times n}$ such that each component of $T_1 x$ and $T_1 y$ is non-zero. Now it is simply a matter of repeating this step for the remaining vectors u and v to complete the proof of the lemma. \square

Proof of Lemma 2.6

We can assume that all components of x, y, u, v are non-zero. To see why this is so, suppose that the result was proven for this case and we were given four arbitrary non-zero vectors x, y, u, v . We could transform them via a single non-singular transformation T such that each component of Tx, Ty, Tu, Tv was non-zero (Lemma 2.5). Then for all symmetric matrices P

we would have

$$\begin{aligned}
(Tx)^T P(Ty) &= x^T (T^T P T) y \\
&= -k u^T (T^T P T) v \\
&= -k (Tu)^T P(Tv)
\end{aligned}$$

Then $Tx = \alpha Tu$ and thus $x = \alpha u$ or $Tx = \beta Tv$ and $x = \beta v$. So we shall assume that all components of x, y, u, v are non-zero. We write x_i for the i^{th} component of the vector x and p_{ij} denotes the entry in the (i, j) position of the matrix P .

Suppose that x is not a scalar multiple of u to begin with. Then for any index i with $1 \leq i \leq n$, there is some other index j and two non-zero real numbers c_i, c_j such that

$$x_i = c_i u_i, x_j = c_j u_j, c_i \neq c_j \quad (21)$$

Choose one such pair of indices i, j . Equating the coefficients of p_{ii}, p_{jj} and p_{ij} respectively in the identity $x^T P y = -k u^T P v$ yields the following equations.

$$x_i y_i = -k u_i v_i \quad (22)$$

$$x_j y_j = -k u_j v_j \quad (23)$$

$$(x_i y_j + x_j y_i) = -k (u_i v_j + u_j v_i) \quad (24)$$

If we combine (21) with (22) and (23), we find

$$y_i = -\frac{k}{c_i} v_i \quad (25)$$

$$y_j = -\frac{k}{c_j} v_j \quad (26)$$

Using (25), (26), (22), (23) and (24) we find

$$\begin{aligned}
c_i u_i y_j + c_j u_j y_i &= -k(u_i v_j + u_j v_i) \\
\Rightarrow -\frac{c_i k}{c_j} u_i v_j - \frac{c_j k}{c_i} u_j v_i &= -k(u_i v_j + u_j v_i) \\
\Rightarrow u_i v_j \left(\frac{c_j - c_i}{c_j}\right) &= u_j v_i \left(\frac{c_j - c_i}{c_i}\right)
\end{aligned}$$

Recall that $c_i \neq c_j$ so we can divide by $c_j - c_i$ and rearrange terms to get

$$\frac{c_i}{c_j} = \left(\frac{v_i}{v_j}\right)\left(\frac{u_j}{u_i}\right) \quad (27)$$

But using (21) we find

$$\frac{c_i}{c_j} = \left(\frac{x_i}{x_j}\right)\left(\frac{u_j}{u_i}\right) \quad (28)$$

Combining (27) and (28) yields

$$\frac{v_i}{v_j} = \frac{x_i}{x_j} \quad (29)$$

Thus $x_i = cv_i$, $x_j = cv_j$ for some constant c .

Now if we select any other index k with $1 \leq k \leq n$, and write

$$x_k = c_k u_k$$

then c_k must be different to at least one of c_i, c_j . Without loss of generality, we may take it that $c_k \neq c_i$. Then the above argument can be repeated with the indices i and k in place of i and j to yield

$$x_i = cv_i, x_k = cv_k. \quad (30)$$

But this can be done for any index k so we conclude that $x = cv$ for a scalar c . So we have shown that if x is not a scalar multiple of u , then it is a scalar multiple of v .

To complete the proof, note that if $x = \beta v$ for a scalar β then by (22),

$$\beta v_i y_i = -k u_i v_i$$

for all i . Thus $y = -\left(\frac{k}{\beta}\right)u$ as claimed. The same argument will show that if $x = \alpha u$ for a scalar α , then $y = -\left(\frac{k}{\alpha}\right)v$. \square