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GEOMETRY IN THE TRANSITION FROM PRIMARY TO POST-PRIMARY

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1. Introduction

This article is intended as a kind of precursor to the document *Geometry for Post-primary School Mathematics*, which forms Section B, pp. 37–84 of the Mathematics Syllabus for Junior Certicate issued by the National Council for Curriculum and Assessment [16] in the context of Project Maths.

Our purpose is to place that document in the context of an overview of plane geometry, touching on several important pedagogical and historical aspects, in the hope that this will prove useful for teachers.

The main points we want to emphasize are these:

- Geometry is a key part of mathematics.
- Children must pass through different stages in studying geometry.
- Each stage plays an important rôle.
- Care must be taken in managing the transitions.
- Some knowledge of history is useful for teachers.

2. The Main Parts of Mathematics

At present, the NCCA presents the mathematics curriculum in terms of *strands*. For primary level the five strands are labelled Number, Algebra, Shape and Space, Measures, and Data. For secondary level they are (1) data, statistics and probability, (2) geometry and trigonometry, (3) number and measure, (4) algebra and (5) functions. The similarity between the two classifications is part of an attempt to foster continuity across the transition. A further initiative designed to foster this was the publication of a *bridging framework* [15] which provides a dictionary linking terminology used in primary to that used in secondary schools. Initially, the divisions were perhaps intended as much to reflect a more-or-less equal division of teaching and learning effort as much as a division of mathematics into its main areas. It was never intended that the strands would represent watertight divisions of the

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subject, and was generally recognised that there is necessary interaction between them.

An analysis of the secondary curriculum reveals that the division becomes progressively more forced at the more advanced levels. It would also be completely impossible to impose it on third-level studies in mathematics. It is standard among mainstream professional researchers to say that there are three main branches in the tree of mathematics: Algebra, Analysis and Geometry. Some would add Probability and Computation to these, but many others would regard Probability and Computation as two branches of Analysis. However, is is also standard that the connections between the main fields of mathematics are so many that it is actually possible to say that any one of these embraces the whole. There are fields such as Algebraic Geometry, Algebraic Topology and Geometric Analysis which may be regarded as branches of either one of two main branches, and which use fundamental results from both.

The point we make here is that from a bird's-eye viewpoint, Geometry is about one third of mathematics. At the research level, it accounts for a solid proportion of new PhD theses, as may be seen by examining the tables in reports of the AMS annual surveys of new graduates [1]. Its applications are in active areas of fundamental and applied physics, robotics, coding, graphics and other commercially significant areas. So it is important. Our students deserve a sound formation in geometry.

For various reasons, in the past decades many students have emerged from secondary school with a poor opinion of geometry, the result of unfortunate experience with the subject. PISA assessment results also showed relatively mediocre performance by the Irish 15-year-old cohort on problems requiring geometrical skill. This needs to change, and indeed change is mandated by the Project Maths curriculum.

3. Stages

3.1. **Primary stages.** The present Primary Curriculum [19] specifies the study of geometrical shapes in two and three dimensions under the heading of Shape and Space, and of length and area under Measure (with support from Number). This begins right at the start, and is developed further year by year. Students are introduced to simple shapes (triangle, rectangle, circle, semicircle, cube, cuboid, sphere, cylinder, cone) and progressively more complex shapes and properties (isosceles triangle, parallelogram, rhombus, pentagon, hexagon, triangular prism, pyramid, scalene triangle, trapezium, regular hexagon) learn to distinguish them and learn names for them and for their parts

and properties. They make use of suitable materials (blocks, paper and scissors, folded paper, art straws, geoboard, mazes, grids, board games, software, plasticine, prisms, compass, string, tangrams, squared paper) and diagrams and learn to recognize shapes in their environment. Most of the work involves flat, planar shapes, but they also manipulate 3-D shapes and solve problems about them. They learn about measuring lengths, areas, volumes and angles (using a progressively richer number system). They learn how the describe and evaluate spatial relations, give directions, construct and draw 2-D shapes using instruments, subdivide and combine shapes, draw tesselations, construct 3-D polyhedra (by folding nets), and use coordinates. They are encouraged to look for common patterns such as lines of symmetry and the result of counting faces minus edges plus vertices for polyhedral shapes. They learn about parallel lines, and right, acute, obtuse and reflex angles. They explore properties of 2-D shapes, including the angle-sum of a triangle and a quadrilateral, and the ratio of diameter to radius of a circle. The syllabus specifies the linkage of geometry to other areas of the curriculum (motor skills, science, art, physical education and dance, geography) and to aspects of everyday life. This is all very useful, and is appropriate for their ages.

At the end of primary school, children should have acquired most basic geometrical concepts and the language that goes with them. They should be in a position to use their understanding to solve many practical problems.

3.2. **Secondary stages.** When they start post-primary school, students should not be allowed to abandon all this geometrical experience, but should continue to draw on it, solidify and develop their understanding of it, and stay in touch with geometrical ideas on a continual basis.

They have a lot more to learn about geometry. There are in fact two further components needed beyond primary level, corresponding to the two main reasons that further geometrical study is needed: the practical utility of more advanced material and skills, and the rôle of geometry in developing and honing the student's reasoning power.

The case for exposure to rigorous mathematical thought as a preparation for life and for any further studies was well made by John Stuart Mill (quoted in [20]):

> *The value of Mathematical instruction as a preparation for those more difficult investigations (physiology, society, government, &c.) consists in the applicability not of*

> *its doctrines, but of its method. Mathematics will ever remain the most perfect type of the Deductive Method in general* $\cdots$
>
> *These grounds are quite sufficient for deeming mathematical training an indispensible basis of real scientific education, and regarding, with Plato, one who is* ἀγεωμέτρητος[1] *as wanting in one of the most essential qualifications for the successful cultivation of the highest branches of philosophy.*

Geometry is not the only branch of Mathematics that may serve to develop reasoning power, but it has long been used for that purpose, and many consider it well-suited. The geometrical theory expounded in the Elements of Euclid (cf. [8]), rediscovered in the West at the end of the Middle Ages and adopted as the preferred text by the first European universities has been the most popular. It is important for teachers to understand some key points about it:

- It is an abstract theory about space (without matter).
- It was not written to be studied by children.
- It has some logical flaws.
- Mathematicians have figured out various ways to fix these flaws so that the main propositions can be proven from a set of axioms.
- Each such amended theory is called Euclidean Geometry.
- Each such theory is even less suited for children.
- Euclidean geometry is very useful.
- There are other geometrical theories, in which some of the propostions of Euclidean Geometry are false.
- We do not actually know which of these is the best approximation to "real empty space"[2].
- We do know that actual space, with matter, does not fit well with Euclidean Geometry[3], although there is a close correspondence at everyday length scales.
- Abstract geometry has to be simplified, if it is to be used in school to develop reasoning power.

[1]– *ageometretos* , i.e. ignorant of geometry (or, perhaps, unskilled in geometry, or indifferent to geometry). The motto said to have been carved above the entrance to Plato's Academy was: Οὐδεις ἀγεωμέτρητος εἰσίτω – *Let no-one ignorant of geometry enter.*

[2]– another abstraction.

[3]when lines are interpreted as light rays.

- Even when simplified, it is not feasible just to fling children into the abstractions without a careful preparatory stage.

We shall elaborate on some of these points, and comment on the pedagogical implications.

We start with some history.

4. Historical development of geometrical theory

4.1. **The arc of history.** Euclidean Geometry has had a long history. Following on practical studies of shapes, lengths, areas and volumes in the Sumerian and Egyptian civilizations *inter alia,* it started to evolve into a logically-organised science as a result of the efforts of philosophers in Greece c.700-600 BC, who wanted to base knowledge on solid foundations. The basic idea was to identify and define purely geometrical (i.e. non-material) abstractions (point, line, etc.) and also identify uncontroversial starting principles about them, and then to use logic to work out the consequences. Ideally, the building blocks of the theory should be as simple as possible. This simple idea proved extremely effective in practical applications (such as tunnel construction), and gave encouragement. Understanding of geometrical theory evolved gradually ever since, although there were many fallow centuries. Euclid's synthesis of the geometry of his day (about 300 BC) was a major landmark, but after his time many further theorems unknown to him have been discovered, and our understanding of the basic plan of his work has also evolved. René Descartes, in his Discourse on the Method (1637 AD) showed how to link numbers to geometry – in Euclid's books, and up to that point geometrical magnitudes and numerical magnitudes had been considered different species. This created the field of algebraic geometry, and in a sense reduced geometry to arithmetic. However it may come as a surprise that the system of real numbers has been fully understood only since about 1860 AD (thanks to Richard Dedekind). Non-euclidean geometries were discovered early in the 19th century, rubbishing Kant's view that our knowledge of Euclidean geometry is "synthetic a priori", and raising the question whether the real world is Euclidean or not.

4.2. **Deductive reasoning.** A proper understanding of logical deductive systems was only arrived at in the late 1800's, and this prompted Hilbert to produce the first fully-rigorous account of Euclidean geometry, i.e. an account in which all the theorems of Euclid can be proved rigorously from first principles. What is now understood as a mathematical theory, or deductive system, has five components [10]:

(1) Undefined terms
(2) Definitions
(3) Axioms
(4) A system of logic (rules for valid deductions)
(5) Theorems (a term that embraces also propositions, lemmas, and corollaries).

In other words, in a logical system we list up-front the terms and assumptions that we start with, and thereafter proceed by way of definitions and proofs.

4.3. **Definitions.** Definitions are about specifying what we are dealing with. A *definition* identifies a new concept in terms of accepted or known concepts. In practice a definition of a word, symbol or phrase E is a statement that E is to be used as a substitute for F, the latter being a phrase consisting of words and possibly symbols or a compound symbol. We accept ordinary words of the English language in definitions and what is at issue is the meaning of technical mathematical words or phrases. In attempting a definition, there is no progress if the technical words or symbols in F are not all understood at the time of the definition.

The disconcerting feature of this situation is that in any one presentation of a topic there must be a first definition and of its nature that must be in terms of accepted concepts. Thus we must have terms which are accepted without definition, that is there must be *undefined* or *primitive* terms. This might seem to leave us in a hopeless position but it does not, as we are able to, and must, assume properties of the primitive terms and work with those.

There is nothing absolute about this process, as a term which is taken as primitive in one presentation of a topic can very well be a defined term in another presentation of that topic, and vice versa. We need *some* primitive terms to get an approach under way.

4.4. **Proof.** Proof is the way to establish the properties of the concepts that we are dealing with. A *proof* is a finite sequence of statements the first of which is called the *hypothesis*, and the last of which is called the *conclusion*. In this sequence, each statement after the hypothesis must follow logically from one or more statements that have been previously accepted. Logically there would be a vicious circle if the conclusion were used to help establish any statement in the proof.

There is also a disconcerting feature of this, as in any presentation of a topic there must be a first proof. That first proof must be based on some statements which are not proved (at least the hypothesis),

which are in fact properties that are accepted without proof. Thus any presentation of a topic must contain unproved statements; these are called *axioms* or *postulates* and these names are used interchangeably.

Again there is nothing absolute about this, as properties which are taken as axiomatic in one presentation of a topic may be proved in another presentation, and vice versa. But we must have *some* axioms to get an approach under way.

4.5. **Hilbert's system.** In Hilbert's system [11] there are undefined terms such as *point, line, plane, between, congruent*, seven axioms of connection, five axioms of order, an axiom of parallels, six axioms of congruence, and an axiom of continuity, and definitions of terms such as *segment, vertex, side of a line*. The logic used is standard Aristotelian logic.

Notice that this leaves aside completely the question of any relation between this theory and the real world. There are equally satisfactory and equally-consistent[4] theories of various geometries in which some of Euclid's theorems are false.

Incidentally, the main aspects of Euclid's work that needed to be "cleaned up" were (1) the attempt to prove the SAS congruence criterion, Prop. I:4, using superposition, instead of just assuming it; (2) the absence of any postulates about line separation or plane separation, and "betweenness", needed for instance in Prop. I:16; and (3) the absence of any continuity or completeness assumption, already an issue in Prop I:1. The notion gained currency in the 1960's that "Euclid is all wrong" and should just be dumped. The truth is that with a little careful tweaking early on, almost everything stands and the proofs can still be used.

4.6. **Alternative Versions of Euclidean Geometry.** Over the period from c.500 BC to the present quite a few different approaches to Euclid's theorems have been published. The superabundance of these is one of the major problems that we face now. Hilbert's was the first that was logically watertight and categorical[5]. Before his time, and since, many variants were invented by teachers who wanted to make Euclid more accessible to children. After him, other professional research mathematicians produced complete versions involving different

[4]The consistency of Euclidean geometry cannot be proven. It can be shown that it is consistent if elementary arithmetic is consistent.

[5]– in the sense that any two interpretations (models) of it are essentially the same

undefined terms, definitions and axioms, but of course the same theorems. They were motivated by the desire to have an equivalent system with simpler axioms. For instance, Hilbert's system does not include or use the real number system, and Birkhoff [3] proposed a system that extended the theory of the real numbers by adding only four axioms and gave all of Euclidean geometry.

5. Geometry for secondary school

5.1. It should be stated honestly, and faced now that a fully rigorous account of plane Euclidean geometry is too difficult for secondary school. This has been widely understood in academic circles for a very long time. In Mathematics Education circles, it was made explicit in the work of the van Hieles (cf. [5]) in the nineteen-fifties, when they identified five levels at which a person might understand geometry, ranging from level zero ("visualization") up to level four ("rigour"). The top level is only appropriate for university-level work. This does not mean that logical work in geometry is not feasible in school – the van Hiele levels appropriate for school were labelled "analysis", "informal deduction" and "formal deduction". Moreover, competence at the top level is not really needed for working with the manifold applications of geometry[6].

A key point is that you cannot train someone in logical deductive thinking by using an illogical system. So what professional mathematicians urge and press for is that school geometry should be in the the context of some fully-rigorous scholarly background approach. The school version should broadly have the same sequence of topics and the same type of proofs, but leave out some very difficult parts, the latter to be guaranteed by those at a higher level who choose to immerse themselves in a study of this material.

The present situation in Ireland is that the prescribed school geometry has for its scholarly background approach the one laid out in [2]. That system, like Birkhoff's, uses the real number system, but employs a few more axioms, including Playfair's version of the parallel axiom [2, Axiom A7, p.57]. The school system is deliberately simplified, as explained in [16, pp.40-43]. As a result, the proofs are not fully watertight, relying in places on unstated "commonsense notions", and a teacher or student who notices this is encouraged to refer to [2] to satisfy themselves that the gaps can be bridged.

[6]The van Hiele model has had a lot of influence. Early on, it formed the basis for a radical reform in the geometry curriculum in the Soviet Union in the nineteen-sixties, and it has gradually been taken on board in the USA.

An important aspect of Birkhoff-like systems is that one can treat the real number system informally (instead of formally, axiomatically) in school. This avoids explicitly mentioning the topics of continuity and completeness, which are too sophisticated for school. An important reason for basing the system on the book [2] is that there the complete scholarly treatment is fully laid out, *with complete detailed proofs*. A number of good alternatives are backed by complete theories for which the full proofs can be generated easily by any competent professional mathematical researcher, but are in fact only sketched in published sources.

5.2. Approaching abstraction when teaching geometry. Euclidean geometry employs abstractions. Right at the start, we have point and line, for instance.

Students have to be prepared carefully for this abstraction. The geometrical concepts must be motivated from the real world around us. Education in geometry (as in everything else) must proceed in stages, as the child's mind develops. These stages have long been recognised, and were explicitly catered for in popular textbooks such as Durrell [7]. (Clement Durell's texts were in widespread use wherever British influence acted from 1919 for over forty years.) There has to be a preliminary stage before the stage of formal logical work with the abstract ideas and it is essential that these stages not be confused with each other. The preliminary stage should not be rushed, and time allowed for the abstract concept to sink in. It is not appropriate to plunge into "Theorem 1" immediately after explaining about points.

Later, when abstract results are applied, we should make it clear that we are now assuming they apply to reality.

A point is not a real thing. It has no size. Durell [7] says that teachers should *never* allow points to be drawn as blobs, and instead indicated by a cross made with two very fine lines. He insists that compass punctures should be as small as possible, and straight lines be as fine as possible. This is extreme, but you see his point! There may well be students who think points are little black round things, as drawn by Geogebra, and it is a good idea to make sure that they are disabused of this before they get started on formal work.

Diagrams are vital in teaching geometry, and should always be used. It is precisely because such visual aids are there to support and guide reason that geometry is considered the best way to practise logic.

There should be considerable physical motivation to start with, and diagrams always used to provide insight, but details of motivation

should not be confused with the careful logical presentation of the mathematical model that follows later on.

Every opportunity should be taken to get students to engage with problems that they can tackle using their current understanding of geometry.

As with any mathematics teaching, one proceeds in a cycle [7, 5]: oral discussion of examples, exercises in numerical and non-numerical examples, informal proof ideas, formal proofs, exercises involving "riders", or "cuts" (extra propositions to be proven by the student – the "Propositions" given without proof in the syllabus document are intended to be used in this way, and it is expected that the assessment process will examine skill in creating such proofs), and one provides exercises graduated by difficulty, extra exercises of one kind for students who struggle, and challenging extra exercises for those pupils "who run ahead of the class". Regarding the latter, although the main focus of the school programme is on plane geometry, one should look out for applications to solid geometry.

6. Lines and Non-Euclidean Geometry

The modern mathematical concept of *line* is infinite, without ends, and is straight. The English word line is derived from the Latin *linea*, which originally referred to flaxen thread, as is the name of the material *linen*, also made from flax. Similarly, in Irish we have the pair of words *líne* and *línéadach*. In Greek, the word for line was γραμμή (gramme), the stroke of a pen, derived from γραφω (grapho), I write, or draw. In contrast to modern usage the Greeks spoke of a *straight line* (literally εὐθεῖα γραμμή , "right line") and *curved line*. Moreover, by straight lines the Greeks mainly meant what we call *line-segments* which would be *produced* (i.e. extended) as required. It is helpful to bear this in mind when reading older texts.

One motivation of a line-segment was a *linen thread held taut*[7]. The notion of being straight was extended to lines, as a segment was unendingly produced, and at each stage there had to be a segment which contained the starting segment. Of course a taut thread could be copied onto a wax, papyrus or wooden tablet, and tablets with straight edges could be cut from the latter. The use of compasses then enabled them to cut out shapes of triangles and various types of quadrilaterals, as well of course as circles. Nowadays we have rulers, protractors, set

[7]Heath[8] records the classical phrase έπ ἀκρον τεταμένη γραμμή – a line stretched to the utmost.

squares and computer software packages to help us draw these figures and make constructions.

Euclid defined a straight line as "a line that lies evenly with the points on itself". This suggests that the concept is an abstraction of the idea of a "line of sight", or if you prefer, of a light ray, although Euclid in the Elements is careful not to refer to any physical thing, perhaps because as a good Platonist he is treating geometry as the form of space, a purer thing than space (cf. [8, pp. 165-169]).

If straight lines are the paths of light rays from an object to our eye, then we now know, from observational Astronomy, that real space is not Euclidean: there are distant objects that can be seen in two different directions (– usually explained in terms of the theory of General Relativity as the result of "gravitational lensing"). So the most that might be true is that real space is approximately Euclidean at some length-scales and where the density of matter is low. Gauss realized that space might be non-Euclidean before most people, and checked measurements taken of the angle-sum in large terrestrial triangles (with vertices on mountain-tops in the Harz) as part of his geodetic survey of the kingdom of Hanover in the early 1820's. These measurements did not show a deviation from 180°, within experimental error. In certain non-Euclidean geometries ("hyperbolic") the angle-sum is always less than 180°, and the defect grows larger and larger as the triangle gets larger – in extremely large triangles the sum may be arbitrarily small! The defect is small in small triangles[8]. It may be that if we could measure the angles in a triangle with vertices in three different galaxies (– hard to see how to do this without visiting them), we would find it substantially less than 180°. We don't know. So we are not justified in instructing our students to "discover that the angle-sum is 180°". They can and should discover that it appears to be very close to this, but that is all. In a class of students, measuring hand-drawn triangles to half a degree, one might expect a range of measurements clustering around 180°, and that is fine.

The proof of the angle-sum theorem, Euclid I:32, [16, Theorem 4], just shows that if you accept the Axioms, then the theorem holds. It is probably not advisable to disturb the faith of the young, but it is as well for teachers to understand this. It hinges on the Axiom of Parallels, and it is amusing to list some of the alternative assumptions that could be used instead (along with the other axioms[9]) , and that

[8]In other non-Euclidean geometries ("elliptic", or "spherical") the angle-sum is consistently greater than 180°

[9]– strictly, the other axioms of "neutral geometry" [10]

are false if it fails (the names in brackets after each are associated with them):

- There are two similar (equiangular) triangles in which each side of one is twice the corresponding side of the other (Wallis, 1663)
- There is at least one rectangle (Saccheri, 1773; Omar Khayyam, 11th Century).
- There is a triangle having area as great as you please (Gauss, 1799).
- A line perpendicular to the bisector of an acute angle at a point inside the angle must meet both arms of the angle.
- A line that cuts one of two parallel lines must cut the other (Proclus, 5th Century AD).

An internet search for the parallel postulate will throw up many more such oddities. The fact that there are so many very plausible statements that imply the Parallel Postulate explains why so many eminent mathematicians were deceived into thinking they had proved it without assuming anything[10].

This explains why there is no such thing as a "physical proof" of a mathematical theorem, and why no theorem is "visually obvious". Any attempt to deduce a mathematical fact from a real-world observation involves a logical gap: the implicit assumption that some mathematical theory accurately reflects some physical reality. If such an assumption is made explicit, it becomes a scientific hypothesis, and can only be disproved by observation, never confirmed [18].

It does not seem to be widely appreciated that, from the logical point of view, the abstract results are also needed in order to lay a firm ground for trigonometry and for coordinate geometry. If the parallel postulate

[10]Courses in non-Euclidean geometry are quite usual in the preparation of mathematics teachers these days. To quote Wolfe [21]: "For teachers and prospective teachers of geometry in the secondary schools the study of Non-Euclidean Geometry is invaluable. Without it there is strong likelihood that they will not understand the real nature of the subject they are teaching and the import of its applications to the interpretation of physical space." He in turn quotes Chrystal, who published a small book about what he called pan-geometry in 1880, aimed at teachers. He wrote: "It will not be supposed that I advocate the introduction of pan-geometry as a school subject; it is for the teacher that I advocate such a study. It is a great mistake to suppose that it is sufficient for the teacher of an elementary subject to be just ahead of his pupils. No one can be a good elementary teacher who cannot handle his subject with the grasp of a master. Geometrical insight and wealth of geometrical ideas, either natural or acquired, are essential to a good teacher of geometry; and I know of no better way of cultivating them than by studying pan-geometry."

fails, we have no rectangles, hence no rectangular cartesian coordinates, and triangles are not similar unless congruent, so we have no standard trigonometry. The theorems about ratios have recently been restored to the Leaving Certificate program, in part because they provide this foundation (retrospectively) for Junior Certificate trigonometry and coordinate geometry, and in part because it was considered important that senior students continue to engage with formal proofs in synthetic geometry.

There may be nothing in nature that corresponds exactly to Euclidean geometry, but it cannot be denied that it has been extraordinarily useful in practical matters for over two and a half millennia. It should also be noted that even if it does not fit exactly the shape of the real universe, Euclidean space, *as an ideal mental construct invented by us*, is immensely useful in other areas of pure and applied mathematics[11] and will always be used. By the way, the useful software system Geogebra is a realization of this ideal mental construct. In it the angle sum is always *exactly* 180°!

7. More History: a fork in the road

We now say a bit about the history of education in geometry.

7.1. Euclid's *Elements* had a virtual monopoly as a textbook for geometry for a very long time. A substantial splinter-group was started in France in the 16th century when Pierre de la Ramée (in Latin Petrus Ramus)(1515-1572), among his other publications, attacked the logical approach of the *Elements*. His views attracted considerable support in French educational circles for many generations and led to widely-held views that what is visually obvious should be accepted without proof. All this led to quite a different approach to geometry with many innovations. Some of these were reflex angles and rotations. Many of these ideas accumulated eventually to efficient new approaches to geometry, in modern times e.g to treatments based on transformations on the one hand and to vector spaces on the other. Thus there are now several possible approaches to Euclidean geometry available but, for present purposes, consideration of them should not be confined to their abstract merits but to which are the most suitable for our school students to obtain their grounding in geometry.

[11]Examples are the theory of equations, numerical computation, much of real and complex analysis, and even non-Euclidean geometry, which is studied using coordinates ("charts") that map pieces of the space to Euclidean space.

7.2. The lead of France was generally followed on the continent of Europe and the notable large country standing out against this and adhering to Euclid's approach was Great Britain. There there were efforts from 1860 onwards to assimilate elements then current on the Continent. The final bulwark to modifying Euclid's *Elements* fell in 1903 with recommendations of the Mathematical Association, and the 'Cambridge Schedule' proposed and adopted at the University of Cambridge about that time. Subsequently in textbooks there, and used in Ireland, a variety of approaches and concepts were mingled from different strands [12, 13, 14, 17]

7.3. This intermingling leads to some severe technical problems. All the textbooks started roughly the same way, focusing on the concepts in the world about us, becoming familiar with shapes and sizes and being led to properties of classes of them. On the whole they were clinging to a bad old habit from Euclid of trying to define everything.

From what we have said above it should be clear that they should have been motivating concepts from the real world, but that in formal geometry undefined but named items are needed. From Euclid they retained the concepts of assumed axioms, or postulates.

The practical difficulty in this is that for some concepts which are obvious and readily understood visually, it is quite difficult to lay down assumed properties for them which lead to their being singled out uniquely. In the next section we deal with two concepts which are at the heart of this problem.

By the way, it is a mistake to think that our forebears were not aware of the need for undefined terms. In books written for undergraduates, as opposed to school children, they express themselves frankly on the point. When Maynooth College was set up in 1795, it was initially staffed by French clerical academics, refugees from the Revolution. Mathematical instruction was compulsory for all students, lay and clerical, and it appears that geometrical teaching was based on French models so that practice at Maynooth was in step with the continental, rather than the British norm. André Darré, first Professor of Mathematics and Natural Philosophy, and formerly of Toulouse, prepared a text in English [6] on plane and spherical geometry for use in the College. He first gives the following definitions:

> *A straight line is that of which the elementary parts run in the same direction. A line is curved, the elements of which change continually their direction.*

Then he says:

> *Such is the most accurate notion Geometry can give of its object; and it is adequate to its object, though not perhaps a logical definition.*
>
> *Sciences mostly begin by such simple ideas sufficiently clear, independently of a definition; and they are no less reasonable than self-evident.*
>
> *For want of such simple notions and self-evident principles, an interminable series of definitions and demonstrations should be required; our mind could find no ground whereon to rest in analysis, or wherefrom to step in synthesis; nothing could be accurately understood, nothing rigorously demonstrated; and a full conviction never be obtained in the pursuit of sciences.*

The organisation of Darré's text leaves something to be desired, even apart from the quality of his English, for which he frankly begs indulgence. He does not give explicit postulates, and his "proof" of the angle-sum theorem for triangles employs a couple of hidden assumptions and a previous result with a useless proof involving motion. Nicholas Callan later wrote what he describes as a revised and improved version of Darré's text, in which he assumes explicit Postulates including a form of the parallel postulate[12]. His definition of parallel[13] is not terribly useful, and is possibly influenced by Legendre [12], two editions of which are in the College Library. But it is clear that he broke with Darré and Legendre in making no attempt to do without an axiom of parallels. Also, Legendre tries to *prove* that all right angles are equal, instead of just assuming it. We suppose that Callan "went back to Euclid", to a large extent.

8. Modern difficulties

8.1. Orientation. Suppose that we draw a triangle and mark a small arrow-head on its boundary to indicate the sense in which we consider a moving point makes one complete circuit of the boundary. Visually it is very clear that there are two possibilities, one of which we name *clockwise* and the other of which we name *anticlockwise*. But how are we to put a definition or properties of that into our mathematical model? If you have drawn your triangle on a sheet of grease-proof paper you will see that the situation is reversed when looked at through the back of the paper, and that is one complication. We cannot put an

[12] "Postulate 4: A straight line that meets one of two parallels, may be produced until it meets the other."

[13] "having no divergency"

arrow-head on each triangle boundary in the plane, so what can we do? Mathematicians have worked out a way to handle this problem by placing an arrow-head on one boundary and improvising a method of transferring that to the boundary of every other triangle in the plane, so that, for example, we can say what is clockwise on every boundary. This concept is named *orientation* of a circuit on a triangle-boundary.

8.2. Rotation. A second awkward topic is the very familiar one of a *rotation* in the real world, or to put it more precisely, 'rotation about a given point, through a given angle'. This involves the difficulty of orientation too. Mathematics has long had a formula for this in coordinate geometry, which uses trigonometry, but how can it be handled if it is introduced early on in pure geometry?

8.3. How should these difficulties be handled? Our position is not that these are very difficult concepts but that they should wait until it is much easier to introduce them. The concepts should be made clear visually by diagrams.

For example orientation, which is rarely dealt with formally, can easily be handled (if someone wants to do that) by using the concept of sensed or signed area in coordinate geometry.

The syllabus documents as they stand do not include a formal treatment of rotations, but do mention them. If a formal treatment were to be added within the existing framework of five strands, a rotation would be a type of function $P \to R(P)$. It would take points P of the plane as inputs, and give other points $Q = R(P)$ as outputs. The relationship between the input point P, the centre of rotation C, and the output point Q would be described in terms of angles and congruence: the angles POQ would all be congruent, and $|PC| = |QC|$.
(A document in which a treatment of this kind is presented as material for a group project may be downloaded from either of the sites
`http://www.ucc.ie/en/euclid/edu_and_careers/projectmaths/` or
`http://archive.maths.nuim.ie/staff/aof/school.html`. These sites also have a few other resources related to school geometry, which have been submitted for approval to the NCCA Project Maths coordinators.)

We note that the text-book of Hall and Stevens [14], very commonly used in Ireland in the past, contained alternative proofs by rotation of some theorems, starting as early as Theorem 1. Teachers may be familiar with these. It may seem attractive to use such proofs, however, the point of training in deductive thinking is lost if proofs can pull in extra axioms out of the blue, and there are no axioms about rotations in the present system. Formal proofs studied should remain within whatever logical framework is laid down in the syllabus, and this in its

turn must be based upon a scholarly back-ground published treatment which provides a context for it.

8.4. Angles and Rotations. This is not to suggest that students should not be made very familiar with the visual concepts of clockwise and anticlockwise rotations in the real world. Of course they should. According to the Primary Curriculum, students are to 'learn to recognise an angle in terms of a rotation' [19, p.75]. This is a bit ambiguous, but perhaps ok. In the formal material on geometry, angle is an undefined term, an abstraction like point and line, so the question is: how is the student to be prepared for this, in the preliminary stage? From this stage, the student will bring some intuitive idea of what an angle is.

It says in the syllabus document that to each angle is associated a unique point called its vertex, two rays starting at the vertex, called its arms, and a piece of the plane called its inside.

It is not going to work very well if the student thinks that an angle is "a rotation". This carries with it some idea of motion, and this is not helpful in studying the angles of, for example, a given triangle. So it would be better, when talking informally about rotations, to say that a rotation is something that can be described in terms of an angle, rather than saying that an angle is a rotation. An angle is a specific "static" object, with vertex, arms and inside.

There are two things: the angle, and the number of degrees in the angle (also known as the measure of the angle [2]). One is a geometrical object, the other is a real number. These can be confused. We suggest that it is a good idea, in the preliminary informal work, to draw various angles, point out the vertex, arms, inside of each, and say that the number of degrees tells us "how big the opening is", "how rapidly the arms diverge", "how much we have to **rotate** one arm about the vertex, in order to reach the other arm", "the amount of turning involved, if we first face along one arm, and then turn and face along the other", and go onto discuss how much of a circle about the vertex is inside the angle, the concepts of degree (and radian, if desired) and the use of the protractor.

Note. The references below include some old books, long out of print and probably not accessible to most teachers. Happily, most such out-of-copyright books may now be accessed and read online, thanks to the Google books initiative and other archiving efforts such as archive.org. We recommend that teachers, when time allows, take advantage of these resources.

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