Electrically charged finite energy solutions of an $SO(5)$ and an $SU(3)$ Higgs-Chern-Simons–Yang-Mills-Higgs systems in $3 + 1$ dimensions

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Abstract

We study spherically symmetric finite energy solutions of two Higgs-Chern-Simons–Yang-Mills-Higgs (HCS-YMH) models in $3+1$ dimensions, one with gauge group $SO(5)$ and the other with $SU(3)$. The Chern-Simons (CS) densities are defined in terms of both the Yang-Mills (YM) and Higgs fields and the choice of the two gauge groups is made so they do not vanish. The solutions of the $SO(5)$ model carry only electric charge and zero magnetic charge, while the solutions of the $SU(3)$ model are dyons carrying both electric and magnetic charges like the Julia-Zee (JZ) dyon. Unlike the latter however, the electric charge in both models receives an important contribution from the CS dynamics. We pay special attention to the relation between the energies and charges of these solutions. In contrast with the electrically charged JZ dyon of the Yang-Mills-Higgs (YMH) system, whose mass is larger than that of the electrically neutral (magnetic monopole) solutions, the masses of the electrically charged solutions of our HCS-YMH models can be smaller than their electrically neutral counterparts in some parts of the parameter space. To establish this is the main task of this work, which is performed by constructing the HCS-YMH solutions numerically. In the case of the $SU(3)$ HCS-YMH, we have considered the question of angular momentum, and it turns out that it vanishes.

1 Introduction

The main task of the present work is to establish that introducing CS dynamics to the YMH system can result in the lowering of the energy of the electrically neutral solution, by giving it electric charge. We have tested this with two distinct models, one with gauge group $SO(5)$ and the other $SU(3)$. The main difference between these two models is that, while the solutions of the $SU(3)$ model considered here carry magnetic charge, those of the $SO(5)$ model have zero magnetic charge.

The two CS densities in $3 + 1$ dimensions employed here and in the preceding work [1] are the first two in an infinite hierarchy, each resulting from the descent [2, 3] from a Chern-Pontryagin density in $2N$ ($N \geq 3$) dimensions. We refer to these as Higgs-CS (HCS) densities. They extend the definition of the usual [4, 5] CS densities to all odd and even dimensions, at the cost of importing a Higgs field. In $2 + 1$ dimensions, it was found [6, 7, 8] that the presence of the (usual) CS density in a gauged Higgs system results in finite energy electrically charged solutions. Here, the corresponding question is considered in $3 + 1$ dimensions. It turns out that different choices of the HCS density employed, result in qualitatively quite different solutions.

The $SO(5)$ HCS-YMH model considered here is that employed in a preceding work [1]. In that preliminary work however, the energy of these solutions increased with the electric charge, and the lowest energy solutions turned out to be those with vanishing charge. In this respect, they are qualitatively similar to JZ solitons [9]. There the electrically neutral solutions had non-vanishing electric YM connection A_0, exhibiting dipole behaviour.

In the present paper we construct more general electrically charged solutions to this $SO(5)$ HCS-YMH model, some of which have lower energy than their neutral counterparts, their energies decreasing with increasing charge. These qualitative features contrast with those of the JZ dyons. As in [1], there are also electrically neutral solutions exhibiting dipole behaviour, namely supporting electrically neutral solutions with non-vanishing electric component of the YM connection A_0. We have constructed three different families of solutions exhibiting these properties, which we refer to as Types I, II, and III. Types I and II describe electrically charged solutions, while Type III solutions describe electrically neutral solutions with non-vanishing electric component A_0 of the YM potential. None of these three types of solutions carry nonzero magnetic charge.

In addition to the $SO(5)$ HCS-YMH model, we have studied an $SU(3)$ HCS-YMH model. The main difference of the $SU(3)$ model is that its solutions carry nonzero magnetic charge, at the same time supporting nonvanishing HCS terms. The resulting electrically charged solitons are dyons which differ fundamentally from the JZ dyon. The feature of decreasing mass with increasing electrical charge, observed for the solutions of the $SO(5)$ HCS-YMH model, persists also for the $SU(3)$ HCS-YMH model. In addition, we have considered the question of angular momentum in the $SU(3)$ case.

The paper is organised as follows. In Section **2** we define the model, which is formally the same for both the $SO(5)$ and $SU(3)$ HCS-YMH models, except for the Higgs symmetry breaking potentials, which are stated there. Symmetry imposition on the respective $SO(5)$ and $SU(3)$ HCS-YMH models is presented in Sections **3** and **4** respectively. In subsections of Sections **3** and **4**, the numerical solutions are presented. Another subsection of Section **4** deals with the question of angular momentum. Finally summary and discussion of our results are given in Section **5**.

2   The models, equations, and charges

The full Lagrangian density is

$$\mathcal{L} = \mathcal{L}_{\text{YMH}} + \kappa_1 \, \Omega_{\text{CS}}^{(1)} + \kappa_2 \, \Omega_{\text{CS}}^{(2)} \,, \tag{1}$$

with the two HCS densities $\Omega_{\text{CS}}^{(1)}$ and $\Omega_{\text{CS}}^{(2)}$ given by

$$\Omega_{\text{CS}}^{(1)} = i \, \varepsilon^{\mu\nu\rho\sigma} \text{Tr} \, \Phi \, F_{\mu\nu} \, F_{\rho\sigma} \,, \tag{2}$$

$$\Omega_{\text{CS}}^{(2)} = i \, \varepsilon^{\mu\nu\rho\sigma} \text{Tr} \left[\Phi \left(\eta^2 \, F_{\mu\nu} F_{\rho\sigma} + \frac{2}{9} \, \Phi^2 \, F_{\mu\nu} F_{\rho\sigma} + \frac{1}{9} \, F_{\mu\nu} \Phi^2 F_{\rho\sigma} \right) \right.$$
$$\left. - \frac{2}{9} \left(\Phi D_\mu \Phi D_\nu \Phi - D_\mu \Phi \Phi D_\nu \Phi + D_\mu \Phi D_\nu \Phi \Phi \right) F_{\rho\sigma} \right] \,, \tag{3}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor in Minkowski spacetime. We do not describe the provenance of the HCS terms Eqs. (2) and (3), since this was given in detail in Appendix **A** of Ref. [1]. The role the Higgs scalar plays here is somewhat akin to that of the axion [10, 11].

The YMH Lagrangian density is [1]

$$\mathcal{L}_{\text{YMH}} = \text{Tr} \left[\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2} \, D_\mu \Phi^2 - \frac{\lambda}{2} \, V[\eta^2, \Phi^2] \right] \,, \tag{4}$$

where $D_\mu = \partial_\mu + [A_\mu, \cdot]$.

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[1]Since we aspire here to present a $3 + 1$ dimensional analogue of the the $2 + 1$ dimensional Chern-Simons-Higgs vortices [7, 8], it may be relevant to inquire whether we could likewise omit the Yang-Mills term in Eq. (4). This in principle is possible since the system excluding the Yang-Mills term is consistent with the Derrick scaling requirement in the corresponding static Hamiltonian after solving for A_0 using the Gauss-Law equation. However in the non-Abelian system at hand, A_0 cannot be solved for in closed form, rendering such an approach impractical.

Here, $V[\eta^2, \Phi^2]$ is the positive definite Higgs selfinteraction potential, with λ its coupling constant, and η denoting the vacuum expectation value of the Higgs field. κ_1 and κ_2 are the coupling strengths of the HCS densities.

The equations of motion resulting from the variations of the Lagrangian with respect to the YM potential and the Higgs field are

$$D_\mu F^{\mu\nu} + [\Phi, D^\nu \Phi] = 2\, i\, \kappa_1\, \varepsilon^{\mu\nu\rho\sigma}\, \{F_{\rho\sigma}, D_\mu \Phi\}\,, \tag{5}$$

$$D_\mu D^\mu \Phi - \lambda \{\Phi, (\Phi^2 + \eta^2\, \mathbb{1})\} = i\, \kappa_1\, \varepsilon^{\mu\nu\rho\sigma}\, F_{\mu\nu}\, F_{\rho\sigma}\,, \tag{6}$$

respectively. $\{\ ,\ \}$ denotes the anticommutator. These equations, Eqs. (5) and (6), are written only for the Lagrangian with $\kappa_2 = 0$ in Eq. (1). This is because the expressions for the right-hand sides of the corresponding equations for $\kappa_2 \neq 0$ are very cumbersome.

There are two types of symmetry breaking potentials consistent with the requirement of finite energy, which we list here for completeness

$$V_1 = \left(\eta^2 + a_1\, \mathrm{Tr}\Phi^2\right)^2\,, \tag{7}$$

$$V_2 = \frac{1}{4}\, \mathrm{Tr}\left(\eta^2\, \mathbb{1} + a_2 \Phi^2\right)^2\,, \tag{8}$$

where the values of a_1 and a_2 will be chosen according to our convenience when imposing symmetries. As it turns out, we will concentrate mainly on $\lambda = 0$ solutions since the presence of the HCS terms, Eqs. (2)-(3), is sufficient to support nontrivial field configurations outside of $SU(2)$. When we do employ a potential for the purpose of checking that our conclusions are not altered by the presence of one, then our choice is Eq. (7) for both the $SO(5)$ and $SU(3)$ models.

The definition of the magnetic monopole charge is

$$\mu = -\frac{1}{4\pi}\, \varepsilon_{ijk} \int_{S^\infty} \mathrm{Tr}\,\Phi\, F_{ij}\, dS_k\,, \tag{9}$$

which presents a lower bound on the energy integral, and the definition of the electric charge is

$$Q = -\frac{1}{4\pi} \int_{S^\infty} \mathrm{Tr}\,\Phi\, F_{i0}\, dS_i\,. \tag{10}$$

The definitions Eqs. (9) and (10) are valid, both when $\kappa_1 = \kappa_2 = 0$, and when $\kappa_1 \neq 0$ and/or $\kappa_2 \neq 0$.

3 Solutions of the $SO(5)$ Higgs-Chern-Simons–Yang-Mills-Higgs model

This Section consists of two Subsections. In Subsection **3.1**, spherical symmetry is imposed and the boundary values of the solutions sought are stated. The numerical construction of the solutions[2] is presented in Subsection **3.2**.

3.1 Imposition of symmetry and boundary values

To proceed to the imposition of symmetry, we note that the fields take their values in the 4×4 chiral Dirac representation of $SO(6)$

$$A_\mu = A_\mu^{\alpha\beta}\, \Sigma_{\alpha\beta}\,, \quad \alpha = i, 4, 5;\ (i = 1, 2, 3)\,, \tag{11}$$

$$\Phi = \psi^{\alpha\beta}\, \Sigma_{\alpha\beta} + \phi^\alpha\, \Sigma_{\alpha 6}\,, \tag{12}$$

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[2]We have employed a collocation method for boundary-value ordinary differential equations, equipped with an adaptive mesh selection procedure [12]. A compactified radial coordinate $x = r/(1+r)$ has been used. Typical mesh sizes include $10^3 - 10^4$ points. The solutions have a relative accuracy of 10^{-8}.

where $(\Sigma_{\alpha\beta}, \Sigma_{\alpha 6})$ are the 4×4 chiral representation matrices of $SO(6)$ [3].

It is convenient to express our Ansatz using the index notation $\alpha = i, M$, $\quad i = 1, 2, 3$, $\quad M = 4, 5$. With this notation, the static spherically symmetric Ansatz for the Higgs field Φ, Eq. (12), and the YM connection $A_\mu = (A_0, A_i)$, Eq. (11), are

$$\Phi = 2\eta \left[\left(\phi^M \Sigma_{M6} + \phi^6 \hat{x}_j \Sigma_{j6} \right) - \left((\varepsilon\psi)^M \hat{x}_j \Sigma_{jM} + \psi^6 \Sigma_{45} \right) \right], \tag{13}$$

$$A_0 = -(\varepsilon\chi)^M \hat{x}_j \Sigma_{jM} - \chi^6 \Sigma_{45}, \tag{14}$$

$$A_i = \left(\frac{\xi^6 + 1}{r} \right) \Sigma_{ij} \hat{x}_j + \left[\left(\frac{\xi^M}{r} \right) (\delta_{ij} - \hat{x}_i \hat{x}_j) + (\varepsilon A_r)^M \hat{x}_i \hat{x}_j \right] \Sigma_{jM} + A_r^6 \hat{x}_i \Sigma_{45}, \tag{15}$$

in which the sum over indices $M, N = 4, 5$ runs over two values such that we can label the functions $(\phi^M, \phi^6) \equiv \vec{\phi}$, $(\chi^M, \chi^6) \equiv \vec{\chi}$, $(\xi^M, \xi^6) \equiv \vec{\xi}$, $(\psi^M, \psi^6) \equiv \vec{\psi}$ and $(A_r^M, A_r^6) \equiv \vec{A}_r$, i.e., in terms of five isotriplets $\vec{\phi}$, $\vec{\chi}$, $\vec{\xi}$, $\vec{\psi}$, and $\vec{A}_r$, all depending on the 3 dimensional spacelike radial variable r. ε being the two dimensional Levi-Civita symbol.

The full one dimensional subsystems are presented in Appendix **A.1**. It immediately follows from Eqs. (13) and (A.1) that the magnetic monopole charge Eq. (9) vanishes.

The important quantity for us here is the global electric charge, Eq. (10), which does not vanish. A straightforward calculation yields the electric field E_i

$$E_i = \mathrm{Tr}\, \Phi\, F_{i0} = -2\eta\, \vec{\psi} \cdot D_r \vec{\chi}, \tag{16}$$

resulting in the electric charge

$$Q = \frac{-1}{4^2\pi} \int_{S^\infty} \mathrm{Tr}\, \Phi\, F_{i0}\, dS_i = \frac{1}{2}\eta \left[r^2 \vec{\psi} \cdot D_r \vec{\chi} \right]_{r=\infty}. \tag{17}$$

For both potentials Eqs. (7) and (8), the finiteness of the energy requires that

$$\lim_{r \to \infty} \left(|\vec{\phi}|^2 + |\vec{\psi}|^2 \right) = 1, \tag{18}$$

so we can introduce an asymptotic angle γ such that

$$\lim_{r \to \infty} |\vec{\phi}|^2 = \cos^2\gamma, \tag{19}$$

$$\lim_{r \to \infty} |\vec{\psi}|^2 = \sin^2\gamma. \tag{20}$$

The $SO(3)$ freedom in this Ansatz results in an invariance at the fixed point of the 2-sphere, due to which only two of the components of each of the five triplets $(\vec{A}_r, \vec{\xi}, \vec{\chi}, \vec{\psi}, \vec{\phi})$ are independent functions. We thus end up with 10 equations of motion for the functions of r,

$$\vec{A}_r = (\tilde{a}_r, 0, a_r)\ ,\ \ \vec{\xi} = (\tilde{w}, 0, w)\ ,\ \ \vec{\chi} = (\tilde{V}, 0, V)\ ,\ \ \vec{\psi} = (\tilde{h}, 0, h)\ ,\ \ \vec{\phi} = (\tilde{g}, 0, g)\ . \tag{21}$$

The equations of motion arising from the variation of $\vec{A}_r$ result in a pair of constraint equations, since there is no non-trivial curvature pertaining to this connection.

We will study three types of solutions, for which these constraint equations are identically satisfied, such that $\vec{A}_r = \vec{0}$ effectively. These finite energy solutions may have a non-vanishing electric charge and zero magnetic charge. It is straightforward to check that the magnetic charge density in Eq. (9) vanishes identically for the field configuration parametrised by our spherically symmetric Ansatz, Eqs. (13), (14), and (15).

These three types of zero magnetic charge solutions are described by the following functions

$$\vec{\xi} = (0, 0, w)\ ,\ \ \vec{\chi} = (\tilde{V}, 0, 0)\ ,\ \ \vec{\psi} = (\tilde{h}, 0, 0)\ ,\ \ \vec{\phi} = (\tilde{g}, 0, 0)\ , \tag{22}$$

$$\vec{\xi} = (0, 0, w)\ ,\ \ \vec{\chi} = (0, 0, V)\ ,\ \ \vec{\psi} = (0, 0, h)\ ,\ \ \vec{\phi} = (0, 0, g)\ , \tag{23}$$

$$\vec{\xi} = (0, 0, w)\ ,\ \ \vec{\chi} = (0, 0, V)\ ,\ \ \vec{\psi} = (\tilde{h}, 0, 0)\ ,\ \ \vec{\phi} = (0, 0, g)\ , \tag{24}$$

to which we refer as Types I, II, and III, respectively. Such solutions exist for models with either of the Higgs potentials, Eqs. (7) and (8).

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[3] The chiral Dirac representation matrices $\Sigma_{\mu\nu} = (\Sigma_{\alpha\beta}, \Sigma_{\alpha 6})$ used here are defined as $\Sigma_{\mu\nu} = -\frac{1}{4}\Sigma_{[\mu} \tilde{\Sigma}_{\nu]}$, in terms of the spin matrices $\Sigma_i = -\tilde{\Sigma}_i = i\gamma_i$, $\Sigma_4 = -\tilde{\Sigma}_4 = i\gamma_4$, $\Sigma_5 = -\tilde{\Sigma}_5 = i\gamma_5$, $\Sigma_6 = +\tilde{\Sigma}_6 = \mathbb{I}$, where $(\gamma_i, \gamma_4, \gamma_5)$, $i = 1, 2, 3$ are the usual Dirac gamma matrices in four dimensions.

3.2 Types I, II, and III: Numerical results

We have not been able to generate numerically excited solutions when all the components in the multiplets Eq. (21) are present. Only solutions for the restricted cases Eqs. (22)-(24) could be found[4].

3.2.1 Type I solutions

These solutions are characterized by $\tilde{w} = 0$, $V = 0$, $h = 0$, and $g = 0$. The expansions at the origin are

$$w = -1 + w_2 x^2 + 2w_2 x^3 + O(x^4) \,, \tag{25}$$
$$\tilde{V} = \tilde{V}_1 x + \tilde{V}_1 x^2 + O(x^3) \,, \tag{26}$$
$$\tilde{g} = \tilde{g}_0 + O(x^2) \,, \tag{27}$$
$$\tilde{h} = \tilde{h}_1 x + \tilde{h}_1 x^2 + O(x^3) \,, \tag{28}$$

where $x = r/(1 + r)$. The asymptotic values of the functions are

$$w = 0 \,, \tag{29}$$
$$\tilde{V} = \tilde{V}_0 \,, \tag{30}$$
$$\tilde{g} = \cos \gamma \,, \tag{31}$$
$$\tilde{h} = \sin \gamma \,, \tag{32}$$

where $\tilde{V}_0$ and γ are free. $\tilde{V}_0$ controls the contribution to the electric charge Eq. (10) of JZ type, while γ gives rise to another contribution to the electric charge, once the HCS terms are present. Our parameters are: λ, κ_1, κ_2, $\tilde{V}_0$, and γ.

The effect of the JZ parameter $\tilde{V}_0$ is exhibited in Fig. 1. For fixed γ and $\kappa_1 = 0$ and $\kappa_2 = 0$, when varying $\tilde{V}_0$ the electric charge Q changes. In this case an increase in $|Q|$ makes the energy of the solutions E increase. This is the behaviour one would expect. In fact, for vanishing λ the theory may be rescaled and the relation between E and Q becomes independent of γ (they both rescale with $\sin \gamma$).

The situation changes radically when the new CS terms are present. In that case the solution with the lowest energy is not the electrically neutral one, in general. There are regions where the energy is a decreasing function of $|Q|$. Both types of HCS terms give rise to such an effect, although the first one, Eq. (2), requires the presence of a non-vanishing potential (i.e., $\lambda \neq 0$). This is shown in Fig. 2, where we exhibit the energy E versus the electric charge Q for type I solutions with $\tilde{V}_0 = 0$, $\kappa_1 = 1.0$, $\kappa_2 = 0$ and $\lambda = 0.0$, 0.1, and 1.0. Clearly, the solution with the largest energy corresponds to the electrically uncharged one (excluding the vacuum solution).

When both contributions to the electric charge are present, the structure of the solutions gets more complicated: several solutions may exist for the same value of the electric charge. Moreover, the uncharged solutions may not exist for large enough values of $\tilde{V}_0$. This is exemplified in Fig. 3 where no electrically neutral solutions exist for these values of the parameters.

The pattern of solutions may develop a large number of branches in certain regions of the parameter space. In Fig. 4 we present the dependence of the energy E on the electric charge Q for type I solutions with $\tilde{V}_0 = 0.5$, $\kappa_1 = 2.0$, $\kappa_2 = -12$ and $\lambda = 0.0$. We observe that several electrically uncharged solutions exist, none of them having the lowest energy.

3.2.2 Type II solutions

In this case, the solutions are characterized by $\tilde{w} = 0$, $\tilde{V} = 0$, $\tilde{h} = 0$, and $\tilde{g} = 0$. The expansions at the origin are

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[4]We have set $\eta = 1/2$ is our numerical schemes. This choice gives rise to a unit energy for type I solutions with $\lambda = 0$, $\kappa_1 = 0$, $\kappa_2 = 0$, and $\gamma = \pi/2$.
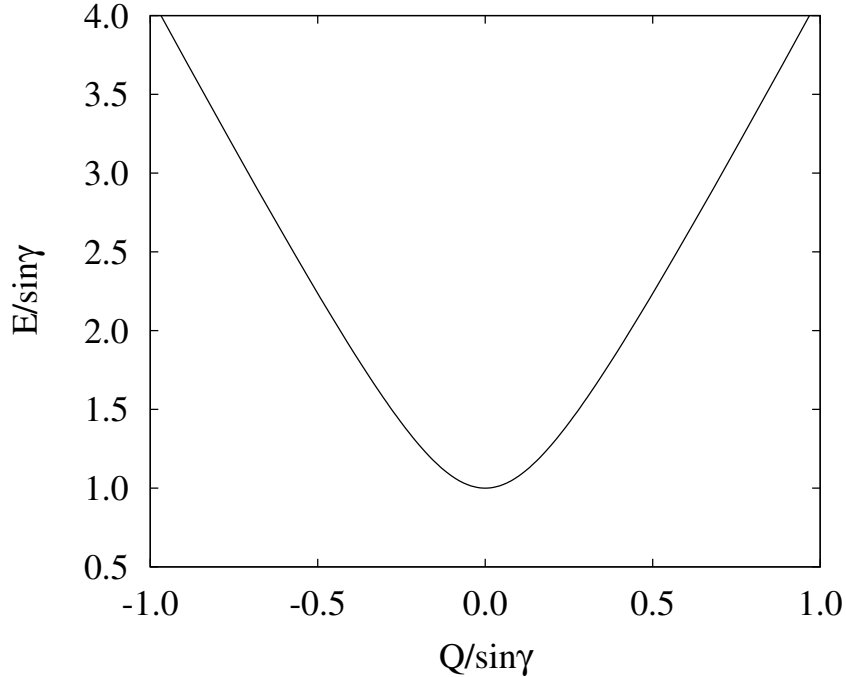
Figure 1: Energy E versus electric charge Q for type I solutions with $\lambda = 0$, $\kappa_1 = 0$, and $\kappa_2 = 0$; $\tilde{V}_0$ is varied and γ is kept fixed.

$$w = -1 + w_2 x^2 + 2w_2 x^3 + O(x^4) \,, \tag{33}$$
$$V = \hat{V}_0 + O(x^2) \,, \tag{34}$$
$$g = g_1 x + g_1 x^2 + O(x^3) \,, \tag{35}$$
$$h = h_0 + O(x^2) \,, \tag{36}$$

where $x = r/(1+r)$. The asymptotic values of the functions are

$$w = 0 \,, \tag{37}$$
$$V = V_0 \,, \tag{38}$$
$$g = \cos\gamma \,, \tag{39}$$
$$h = \sin\gamma \,, \tag{40}$$

where γ is free. V does not enter the equations directly, but just through its derivatives. So the asymptotic value of V, V_0, may be given any arbitrary value (gauge freedom). So for this type of solutions we do not have V_0 as a true physical parameter to be varied; that means there is no JZ parameter. Then, only γ allows us to vary the electric charge of the solutions, once the other parameters of the theory, namely, λ, κ_1, and κ_2, are given.

As opposed to type I solutions, for type II solutions the first HCS term, Eq. (2), can give rise to charged solutions also for $\lambda = 0$. When only one type of the HCS term is present, the structure of the solutions is quite simple, as shown in Fig. 5. When both are present, the structure becomes more involved, although the lack of a JZ term prevents the appearance of very complicated structures as in Fig. 4. In Fig. 6 we show the energy E versus the electric charge Q for $\lambda = 0.0$, $\kappa_1 = 2.0$, and $\kappa_2 = -12.0$. Again, the uncharged solutions (excluding the vacuum) do not correspond to the solutions with the lowest energy.
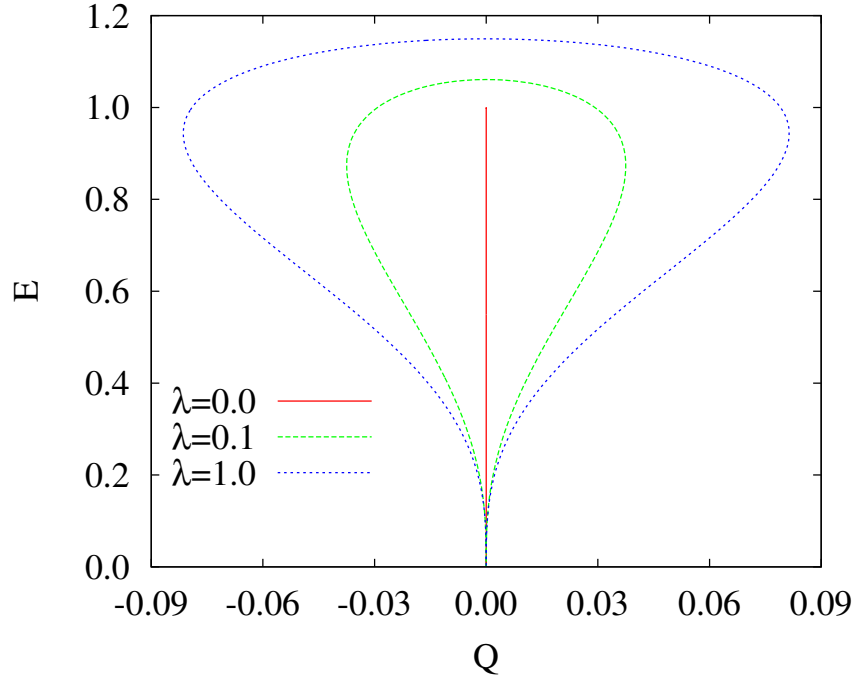
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Figure 2: Energy E versus electric charge Q for type I solutions with $\tilde{V}_0 = 0$, $\kappa_1 = 1.0$, and $\kappa_2 = 0$ for three values of λ: 0.0, 0.1, and 1.0.

3.2.3 Type III solutions

When we set $\tilde{w} = 0$, $\tilde{V} = 0$, $\tilde{g} = 0$, and $h = 0$, type III solutions are obtained. The expansions at the origin now read

$$w = -1 + w_2 x^2 + 2w_2 x^3 + O(x^4)\,, \tag{41}$$
$$V = \hat{V}_0 + O(x^2)\,, \tag{42}$$
$$g = g_1 x + g_1 x^2 + O(x^3)\,, \tag{43}$$
$$\tilde{h} = \tilde{h}_1 x + \tilde{h}_1 x^2 + O(x^3)\,, \tag{44}$$

where $x = r/(1+r)$. The asymptotic values of the functions are

$$w = 0\,, \tag{45}$$
$$V = 0\,, \tag{46}$$
$$g = \cos\gamma\,, \tag{47}$$
$$\tilde{h} = \sin\gamma\,, \tag{48}$$

where γ is free. When the electric charge Q, Eq. (17), is evaluated for these solutions, it is found to be zero. However, the electric potential, A_0, is not identically zero. This is clearly seen in Fig. 7 where the functions w, V, g, and $\tilde{h}$ are shown for the type III solution with $\lambda = 0.0$, $\gamma = 1.2$, $\kappa_1 = 1.0$, and $\kappa_2 = 2.0$.

Since the electric charge vanishes in this case, we may show the structure of branches plotting the energy E versus the asymptotic angle γ. Very intricate patterns appear, as demonstrated in Fig. 8 for the type III solutions with $\lambda = 0.0$, $\kappa_1 = 1.0$, and $\kappa_2 = -12.0$.
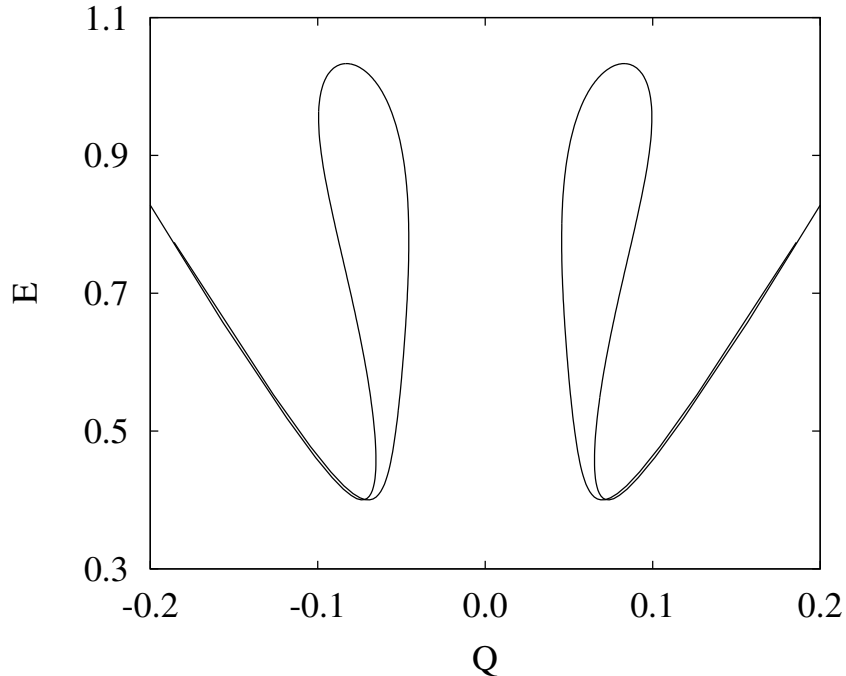
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Figure 3: Energy E versus electric charge Q for type I solutions with $\lambda = 0$, $\tilde{V}_0 = 0.2$, $\kappa_1 = 0.3$, and $\kappa_2 = 0.5$.

4 Solutions of the $SU(3)$ Higgs-Chern-Simons–Yang-Mills-Higgs model

This Section consists of three Subsections. In Subsection **4.1**, spherical symmetry is imposed and the boundary values of the solutions sought are stated. The numerical construction of the solutions is presented in Subsection **4.2**, and in Subsection **4.3** we impose axial symmetry on this system with a view to show whether the dyon of the $SU(3)$ HCS-YMH model rotates or not.

Solutions of $SU(3)$ monopoles have been studied intensively a long time ago [13]. Here we follow the (some of the) constructions to be found in [14] and [15].

While in the previous example, namely the $SO(5)$ model on $\mathbb{R}^{3+2}$, the dimensional descent from 8 (and resly. 6) over S^3 (and resly. S^1) giving rise to $HCS(2)$ (and resly. $HCS(1)$) was that prescribed in [2], here the corresponding prescription is slightly different. Instead of the gauge field in the bulk being a 8×8 (and resly. 4×4) anti-Hermitian connection, here it is a 6×6 (and resly. 3×3) anti-Hermitian connection.

4.1 Imposition of symmetry and boundary values

We use the standard $SU(3)$ spherically symmetric Ansatz

$$A_i = \left(\frac{1-w}{r} \right) \lambda_{ij}^{(1)} \hat{x}_j , \tag{49}$$

$$\Phi = \frac{1}{2} i \eta \left(h \hat{x}_j \lambda_j^{(1)} + g \lambda_8 \right) , \tag{50}$$

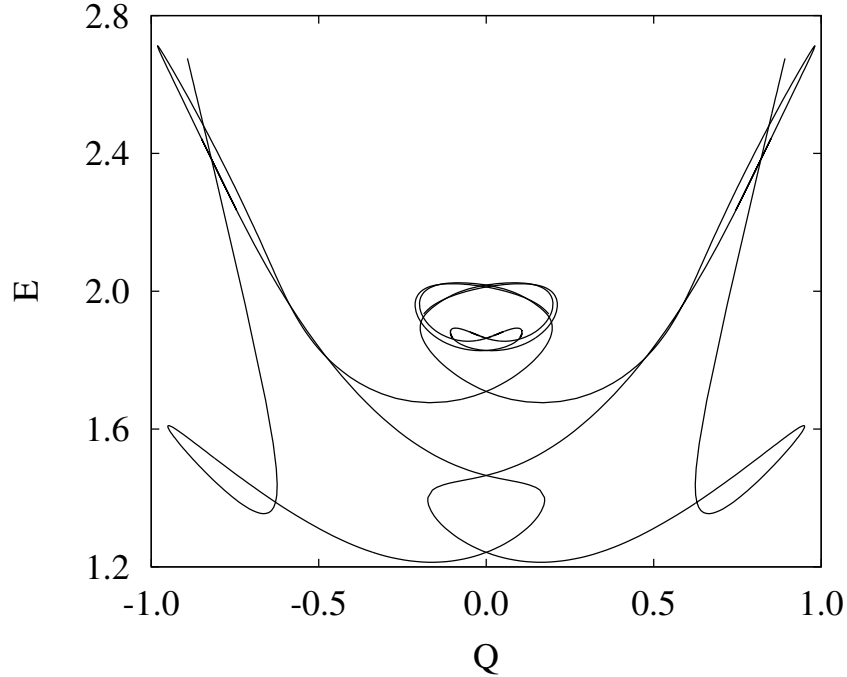$$A_0 = \frac{1}{2} i \left(u \hat{x}_j \lambda_j^{(1)} + v \lambda_8 \right) . \tag{51}$$

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Figure 4: Energy E versus electric charge Q for type I solutions with $\lambda = 0$, $\tilde{V}_0 = 0.5$, $\kappa_1 = 2.0$, and $\kappa_2 = -12.0$.

$\lambda_i^{(1)}$, $i = 1, 2, 3$ are the first three $su(2)$ embeddings in $su(3)$, λ_8 is the last diagonal one, and

$$\lambda_{ij}^{(1)} = -\frac{1}{4}[\lambda_i^{(1)}, \lambda_j^{(1)}].$$

(We have used anti-Hermitian representations of the $su(3)$ algebra.)

Detailed one dimensional reduced quantities used in our computations are given in Appendix **A.2**.

The expansions at the origin read

$$w = -1 + w_2 x^2 + 2w_2 x^3 + O(x^4), \tag{52}$$
$$h = h_1 x + h_1 x^2 + O(x^3), \tag{53}$$
$$g = g_0 + O(x^2), \tag{54}$$
$$u = u_1 x + u_1 x^2 + O(x^3), \tag{55}$$
$$v = v_0 + O(x^2). \tag{56}$$

We seek solutions with the following asymptotic values

$$\lim_{r \to 0} w(r) = 1, \quad \lim_{r \to \infty} w(r) = 0, \tag{57}$$
$$\lim_{r \to 0} h(r) = 0, \quad \lim_{r \to \infty} h(r) = \cos\gamma, \tag{58}$$
$$\lim_{r \to 0} g'(r) = 0, \quad \lim_{r \to \infty} g(r) = \sin\gamma, \tag{59}$$
$$\lim_{r \to 0} u(r) = 0, \quad \lim_{r \to \infty} u(r) = u_0, \tag{60}$$
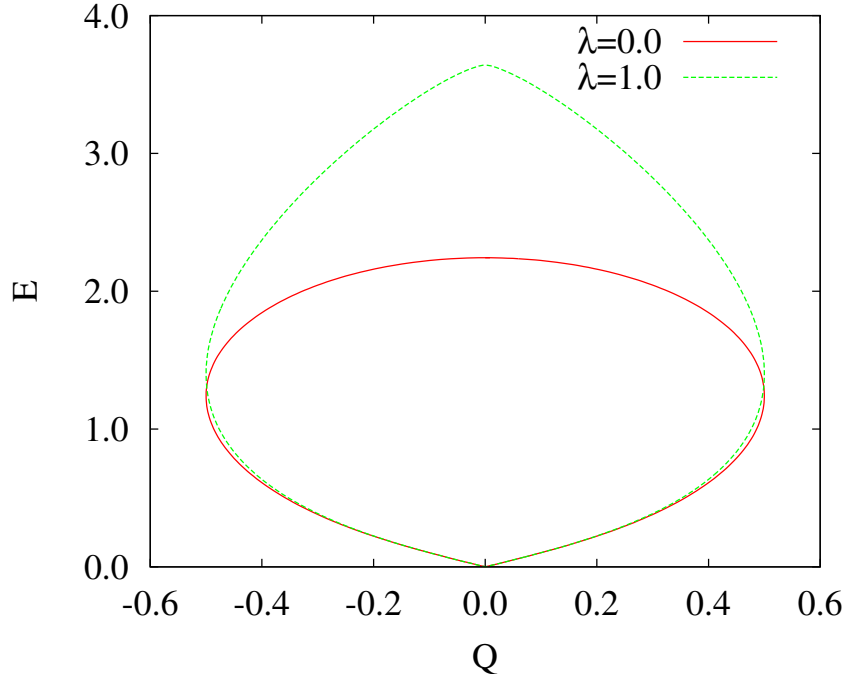$$\lim_{r \to 0} v'(r) = 0, \quad \lim_{r \to \infty} v(r) = 0 \text{ (gauge choice)}, \tag{61}$$

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Figure 5: Energy E versus electric charge Q for type II solutions with $\lambda = 0.0$, 1.0, $\kappa_1 = 1.0$, and $\kappa_2 = 0.0$.

where γ and u_0 are free parameters, corresponding to an asymptotic angle for the Higgs components and the JZ parameter, respectively.

Under these boundary conditions, the magnetic charge, Eq. (9), becomes

$$\mu = \eta \cos \gamma \,, \tag{62}$$

and the electric charge, Eq. (10), results to be

$$Q = \frac{1}{2}\eta \left[r^2 \left(hu' + gv' \right) \right]_{r=\infty} . \tag{63}$$

In the absence of the HCS terms, when $\lambda = 0$ the second-order field equations are solved by the first-order selfduality equations. The latter reduce to the BPS equations which have nontrivial solutions only for the functions $w(r)$, $h(r)$ and $u(r)$, while the functions $g(r)$ and $v(r)$ both vanish everywhere. This means that with $\lambda = 0$ the only solutions are the $SU(2)$ JZ dyons in that case. However, when the HCS terms are present, nontrivial solutions for the functions $g(r)$ and $v(r)$ are present even in the $\lambda = 0$ limit. Since the parameter space is already large enough, we will restrict our attention in this work to the $\lambda = 0$ case only, for economy of presentation.

4.2 Numerical results

We have generated numerical solutions to this theory. In these numerical ruesults we have set $\eta = 1$ to fix the scale.

As happened for $SO(5)$, when $\kappa_1 = 0$, $\kappa_2 = 0$, and $\lambda = 0$ the representation of the scaled energy E/μ versus the scaled electric charge Q/μ shows that E/μ is an increasing function of $|Q/\mu|$; in fact, the figure coincides with Fig. 1 (when rescaled properly). The situation changes, however, when the HCS terms are present. In that case, for a given asymptotic angle γ (i.e., a given magnetic charge μ), the electrically uncharged solution need not be the one with the least energy. We exhibit this fact in Fig. 9 where we represent the energy E versus the magnetic
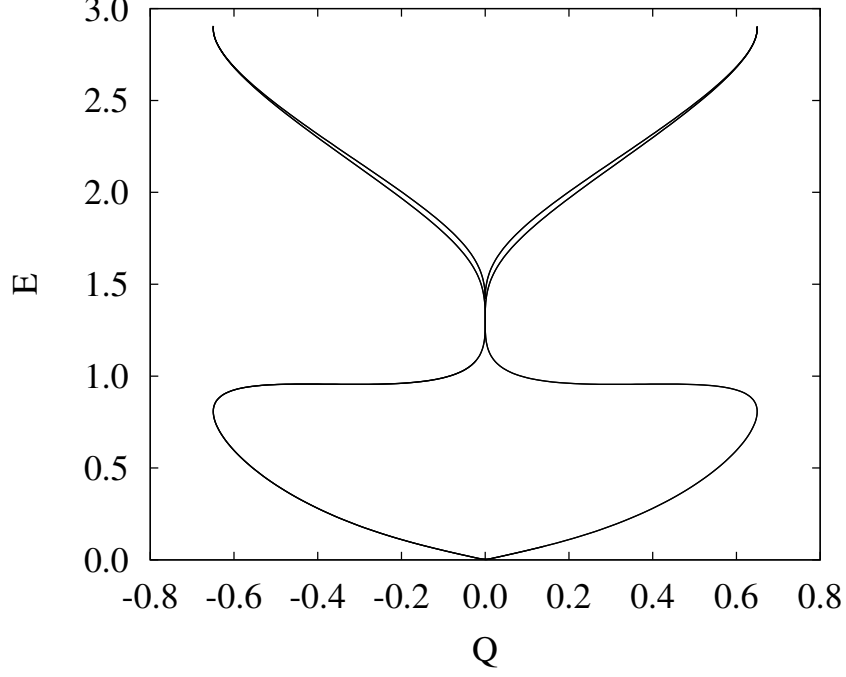
Figure 6: Energy E versus electric charge Q for type II solutions with $\lambda = 0.0$, $\kappa_1 = 2.0$, and $\kappa_2 = -12.0$.

charge μ for $\lambda = 0$, $\kappa_1 = 1$, and $\kappa_2 = 1$ and several values of the electric charge Q: 0.0, 0.5, and 1.0. (Notice that in the limit $\mu = 0$ the value of the energy tends to the value of the electric charge.)

This effect is more clearly observed in Fig. 10, where we represent the energy E of the solutions versus the electric charge Q for 3 asymptotic angles $\gamma = 0$, $\pi/6$, and $\pi/3$ for $\lambda = 0$, $\kappa_1 = 1$, and $\kappa_2 = 1$. For nonvanishing γ the minimal energy occurs for a nonvanishing of the electric charge.

4.3 The issue of angular momentum

The issue of angular momentum density can readily be calculated using the Ansatz given in Eqs. (B.8)-(B.11),

$$4\,T_\varphi{}^0 = [(D_\rho\xi \cdot D_\rho\chi) + (D_z\xi \cdot D_z\chi)] + 4\eta^2\rho\,(\phi\varepsilon\chi)(\phi\varepsilon\xi)\,, \tag{64}$$

which can be rewritten in the form

$$\begin{aligned}
4\,\rho\,T_\varphi{}^0 &= [\partial_\rho(\rho\,\xi \cdot D_\rho\chi) + \partial_z(\rho\,\xi \cdot D_z\chi)] \\
&\quad - [(\xi \cdot D_\rho\chi) + \rho\,\xi \cdot (D_\rho D_\rho\chi + D_z D_z\chi)] - 4\eta^2\rho\,(\phi\varepsilon\chi)(\phi\varepsilon\xi)\,,
\end{aligned} \tag{65}$$

where a total divergence term is isolated.

Consider now the equation resulting from the variation of Eq. (B.27) with respect to the doublet χ^a,

$$\left[D_\rho\chi^a + \rho(\,D_\rho D_\rho\chi^a + D_z D_z\chi^a) - \frac{1}{\rho}(\chi\varepsilon\xi)(\varepsilon\xi)^a \right] + 4\eta^2\rho\,(\phi\varepsilon\chi)(\phi\varepsilon)^a =$$
$$= \frac{1}{2}\,\kappa_1\,\eta\,\left[D_{[\rho}(g\,D_{z]}\xi)^a + g\,f_{\rho z}\,(\varepsilon\xi)^a \right]\,. \tag{66}$$

Contracting Eq. (66) with ξ^a and substituting the result in Eq. (65)

$$4\,\rho\,T_\varphi{}^0 = [\partial_\rho(\rho\,\xi \cdot D_\rho\chi) + \partial_z(\rho\,\xi \cdot D_z\chi)] - \frac{1}{2}\,\kappa_1\,\eta\,\left[\xi \cdot D_{[\rho}(g\,D_{z]}\xi) \right]\,. \tag{67}$$
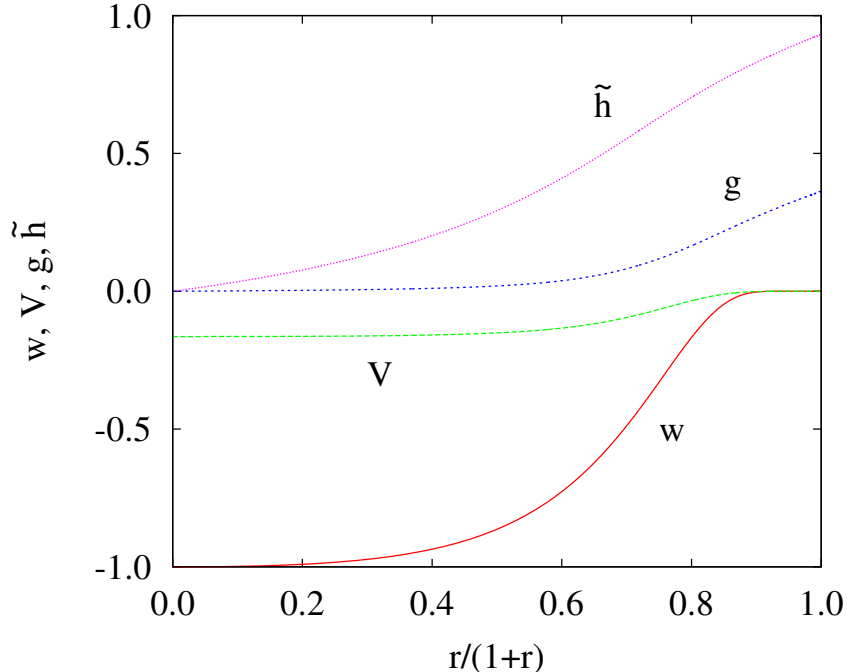
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Figure 7: Functions w, V, g, and $\tilde{h}$ for type III solutions with $\lambda = 0.0$, $\gamma = 1.2$, $\kappa_1 = 1.0$, and $\kappa_2 = 2.0$.

The first term on right-hand side in Eq. (67) is a **div** and its volume integral vanishes by virtue of the asymptotic values of the solutions.

The second term is a **curl**. Using the notation

$$x_A = (\rho, z),$$

the second term in Eq. (67) can be expressed as

$$\begin{aligned}
\xi^a D_{[\rho}(g\, D_{z]}\xi)^a &= \varepsilon_{AB}\, \xi^a\, D_A(g\, D_B\xi)^a \\
&= -\frac{1}{2}\, \varepsilon_{AB}\, \partial_A \left(|\xi|^2\, \partial_B g\right),
\end{aligned}$$

which can be evaluated by performing a contour integral, using Stokes' Theorem (like the multi-monopole charge.)

On the far hemisphere, $|\xi|^2 = 0$ so there will be no contribution. On the z-axis $\partial_z g$ changes sign going through the origin, so the line integral on the positive z-axis will cancel against the line integral on the negative z-axis. Thus, the angular momentum of this system vanishes.

5   Summary, comments and outlook

In this Section, we will summarise our results and comment on their properties. After that we will describe what further questions may arise out of the results. In this paper we have constructed electrically charged solitons in two distinct YMH models in $3 + 1$ dimensions, one with gauge group $SO(5)$ and the other $SU(3)$. Both these theories involve two (dynamical) new CS terms which we refer to as HCS terms. The purpose of this investigation is to show that in certain regions of the parameter space, the electrically charged solutions have smaller mass than their electrically neutral counterparts. This property is a consequence of the dynamics of the HCS densities appearing in the respective Langrangian. This is the main result presented here.
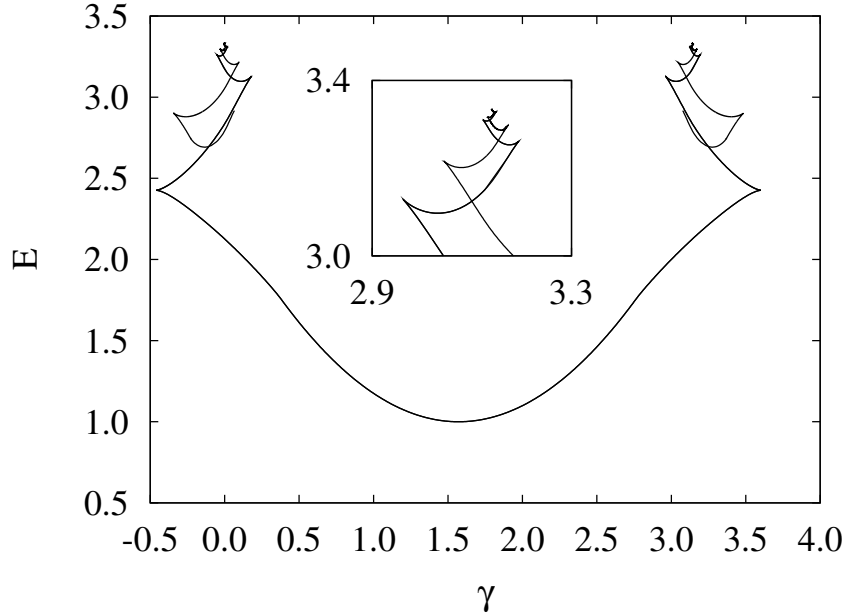
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Figure 8: Energy E versus asymptotic angle γ for type III solutions with $\lambda = 0.0$, $\kappa_1 = 1.0$, and $\kappa_2 = -12.0$.

This investigation is carried out for two distinct models to show that the main result obtained here, is independent of the specific feature of the model chosen, namely of the choice of gauge group. The $SO(5)$ and $SU(3)$ models employed differ in an important respect, namely that the former has zero magnetic charge while the latter has a magnetic charge (in the spherically symmetric case). It is reasonable to treat these two types of solutions separately, to ensure that such a prominent difference does not result in the main feature claimed.

Solutions to the $SO(5)$ and $SU(3)$ models share two properties. First, when the HCS terms are decoupled, *i.e.* setting $\kappa_1 = \kappa_2 = 0$, the energy of the charged soliton increases with increasing electric charge. This expected result is exhibited in Figure. 1.

Another consequence of setting $\kappa_1 = \kappa_2 = 0$ in these models is, that in the absence of the Higgs symmetry breaking potential ($\lambda = 0$) only solutions parametrising the $SO(3)$ subgroup are supported. However, when κ_1 and/or κ_2 are switched on, the gauge fields can take their values outside of $SO(3)$. It is therefore not necessary to consider $\lambda > 0$ solutions and for simplicity we have concentrated on the $\lambda = 0$. We have nonetheless considered $\lambda > 0$ models in a few cases, to ensure that the introduction of the Higgs potential does not alter the qualitative features of our main result.

5.1 The $SO(5)$ model

In this case we have only zero magnetic charge solutions. These exhibit the desired property in some regions of the parameter space. To make our investigation complete, we have studied three types of such solutions, Type I, II and III. The numerical construction of these solutions is presented in Section **3.2**.

- Type I solutions are characterised by the existence of two parameters: one of them related to the JZ contribution to the electric charge, $\tilde{V}_0$, and one related to the HCS contribution, γ. These solutions posses a non-vanishing electric charge coming from both types of sources. Uncharged solutions may have higher energy than the charged ones. These results are exhibited in Figs. 1-4 where we exhibit the dependence of the energy E on the electric charge Q under several circumstances.

- Type II solutions are characterised by the presence of the asymptotic angle γ. In this case there is no JZ
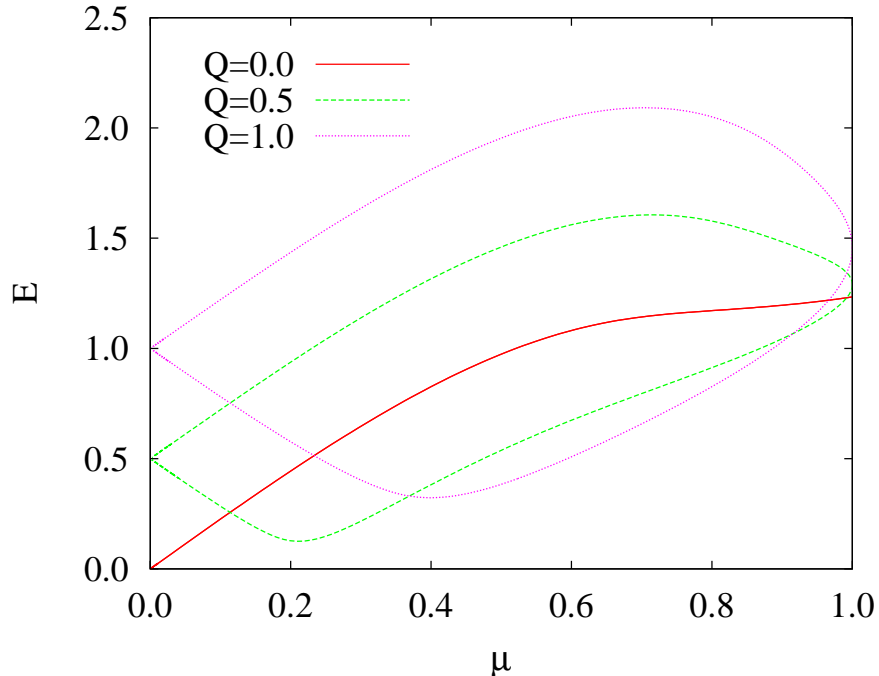
Figure 9: Energy E versus magnetic charge μ for solutions with $\lambda = 0$, $\kappa_1 = 1$, and $\kappa_2 = 1$ and several values of the electric charge Q: 0.0, 0.5, and 1.0.

parameter free. These results are exhibited in Fig. 5 and 6. As for Type I solutions, these solutions are electrically charged and their mass may be lower than that of the uncharged solution.

- Type III solutions are characterised also by the asymptotic angle γ. Opposite to the previous two type these solutions are electrically uncharged although their electric potential is not identically zero. They describe electric dipoles with zero electric monopole. These results are exhibited in Figs. 7 and 8. The structure of these solutions may get quite complicated as shown in Fig. 8.

Note that in Figs 2 and 5, profiles with $\lambda > 0$ appear, which preserve the shapes conformally.

5.2 The $SU(3)$ model

The main feature in this case is that the solutions carry both electric and magnetic charge, and are dyons. We see that the qualitative features observed in the $SO(5)$ model, namely our main result, are preseved. While the qualitative result, that the electrically neutral solutions can be more massive than the neutral ones, a specific feature is observed.

- In Fig. 9 we observe that for non-vanishing electric charge, two dyonic solutions are possible for a magnetic charge $0 < \mu < 1$. The mass of the magnetic monopole (curve in red) is higher than the corresponding value along the lower branch for large ranges in μ. That indicates that the electrically neutral solutions are not necessarily the least energetic ones, in general.

- In Fig. 10 we show this effect more clearly for 3 values of the magnetic charge (including the one chosen in Weinberg's book [15] (green curve)). For $0 < \mu < 1$ the minimum of the energy occurs for non-vanishing electric charge.
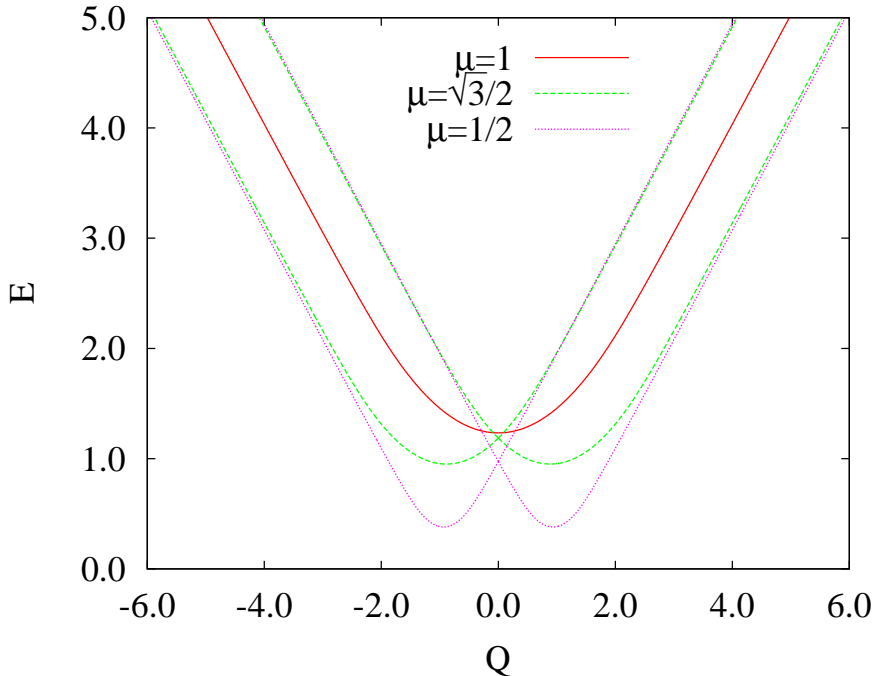
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Figure 10: Energy E versus electry charge Q for solutions with $\lambda = 0$, $\kappa_1 = 1$, and $\kappa_2 = 1$ and several values of the magnetic charge μ: $1, \sqrt{3}/2$, and $1/2$.

In addition in this case we have considered the axially symmetric fields and have constructed the angular momentum density of this $SU(3)$ dyon. It turns out that this vanishes.

5.3 Summary and outlook

In this paper we have constructed electrically charged solitons in an $SO(5)$ and $SU(3)$ HCS-YMH theory in $3+1$ dimensions. These theories contain the new CS terms which were employed in [1] for the $SO(5)$ model. By means of an enlarged spherically symmetric Ansatz, we have been able to endow the solutions of the $SO(5)$ model [1] with an asymptotic angle γ resulting in a larger set of electrically charged solutions, which exhibit the new desired properties. Qualitatively similar results are obtained for the $SU(3)$ model. This way of producing electrically charged solutions differs from the prescription of Julia and Zee [9]. Technically, in the $SO(5)$ model, the obvious difference with the JZ prescription is that the time component of the YM potential A_0 and the Higgs field do not take their values in the same representation of the gauge group. But more importantly, the origin of the electrical fields here is found in the CS dynamics in the case of both the $SO(5)$ and $SU(3)$ models. This is akin to the analogous $2+1$ dimensional situation in [6] and [7, 8].

In the case of the $SU(3)$ model we have calculated the angular momentum of the CS dyon and found that it vanishes. In this respect, the introduction of a new CS term with the attendant enlargement of the gauge group from $SU(2)$ to $SU(3)$, does not change the general result in [16] (and references therein), namely that $SU(2)$ YMH dyons in $3+1$ dimensions do not rotate. This property contrasts with the analogous $2+1$ dimensional situation in [6] and [7, 8], where the introduction of the CS term results in rotation. In the matter of electric charge the introduction of a CS term plays the same role in gauge-Higgs theories in both $3+1$ and $2+1$ dimensions. Thus, the effect of CS dynamics in $3+1$ and $2+1$ dimensions is qualitatively different, overlapping in one respect (electric charge) but differing in another (angular momentum). This question is at present under intensive consideration.

Finally, it is natural to inquire what the analogue of the present investigation in the context of gauged Higgs

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models would be, in the case of gauged Skyrme [17] systems. For this, one would have to employ the Skyrme analogue of the HCS densities used here. This question is at also under intensive consideration.

A   The one dimensional quantities subject to spherical symmetry

In this Appendix, we present the curvature field strengths and the covariant derivatives subject to spherical symmetry. The resulting one dimensional static Lagrangian and energy densities used in our computations are then displayed. These quantities are given in the following two subsections, each for the $SO(5)$ and the $SU(3)$ models, respectively.

A.1 $SO(5)$ model

The parametrisation used in the Ansatz, Eqs. (13)-(15), results in a gauge covariant expression for the YM curvature $F_{\mu\nu} = (F_{ij}, F_{i0})$ and the covariant derivative of the Higgs $D_\mu\Phi = (D_i\Phi, D_0\Phi)$

$$F_{ij} = \frac{1}{r^2}\left(|\vec{\xi}|^2 - 1\right)\Sigma_{ij} + \frac{1}{r}\left[D_r\xi^6 + \frac{1}{r}\left(|\vec{\xi}|^2 - 1\right)\right]\hat{x}_{[i}\Sigma_{j]k}\hat{x}_k + \frac{1}{r}D_r\xi^M\hat{x}_{[i}\Sigma_{j]M}\,, \tag{A.1}$$

$$\begin{aligned}F_{i0} = &-\frac{1}{r}\xi^M(\varepsilon\chi)^M\Sigma_{ij}\hat{x}_j + \frac{1}{r}\left[\xi^6(\varepsilon\chi)^M - \chi^6(\varepsilon\xi)^M\right]\Sigma_{iM}\\ &-\left\{(\varepsilon D_r\chi)^M + \frac{1}{r}\left[\xi^6(\varepsilon\chi)^M - \chi^6(\varepsilon\xi)^M\right]\right\}\hat{x}_i\hat{x}_j\Sigma_{jM} - D_r\chi^6\,\hat{x}_i\,\Sigma_{45}\,, \end{aligned}\tag{A.2}$$

$$\begin{aligned}(2\eta)^{-1}D_i\Phi = &-\frac{1}{r}(\vec{\xi}\cdot\vec{\phi})(\delta_{ij} - \hat{x}_i\hat{x}_j)\Sigma_{j6} + D_r\phi^M\hat{x}_i\,\Sigma_{M6} + D_r\phi^6\,\hat{x}_i\hat{x}_j\,\Sigma_{j6}\\ &-\frac{1}{r}\xi^M(\varepsilon\psi)^M\Sigma_{ij}\hat{x}_j + \frac{1}{r}\left[\xi^6(\varepsilon\psi)^M - \psi^6(\varepsilon\xi)^M\right]\Sigma_{iM}\\ &-\left\{(\varepsilon D_r\psi)^M + \frac{1}{r}\left[\xi^6(\varepsilon\psi)^M - \psi^6(\varepsilon\xi)^M\right]\right\}\hat{x}_i\hat{x}_j\Sigma_{jM} - D_r\psi^6\,\hat{x}_i\,\Sigma_{45}\,, \end{aligned}\tag{A.3}$$

$$\begin{aligned}(2\eta)^{-1}D_0\Phi = &\;\phi^M(\varepsilon\chi)^M\hat{x}_j\,\Sigma_{j6} - \left[\phi^6(\varepsilon\chi)^M - \chi^6(\varepsilon\phi)^M\right]\Sigma_{M6}\\ &+\chi^M\psi^N\Sigma_{MN} - (\psi^6\chi^M - \chi^6\psi^M)\,\hat{x}_j\,\Sigma_{jM}\,, \end{aligned}\tag{A.4}$$

in which we have used the notation

$$D_r\phi^a = \partial_r\phi^a + \varepsilon^{abc}A_r^b\,\phi^c\,,\quad \dots$$

as the $SO(3)$ covariant derivatives of the four triplets $\vec{\xi} \equiv \xi^a = (\xi^M, \xi^6)$, $\vec{\chi} \equiv \chi^a = (\chi^M, \chi^6)$, $\vec{\psi} \equiv \psi^a = (\psi^M, \psi^6)$, and $\vec{\phi} \equiv \phi^a = (\phi^M, \phi^6)$, with respect to the $SO(3)$ gauge connection $\vec{A}_r \equiv A_r^a$.

Substituting Eq. (13) and Eqs. (A.2) in the HCS densities, Eqs. (2)-(3), we have the reduced one dimensional HCS densities

$$\omega_{\mathrm{CS}}^{(i)} \stackrel{\text{def.}}{=} \kappa_i r^2 \Omega_{\mathrm{CS}}^{(i)}\,,\quad i = 1, 2\,, \tag{A.5}$$

where for the first HCS term, Eq. (2), we have the reduced one dimensional density $\omega_{\mathrm{CS}}^{(1)}$,

$$\omega_{\mathrm{CS}}^{(1)} = 8\kappa_1\,\eta\,\left[(|\vec{\xi}|^2 - 1)\,\vec{\phi}\cdot D_r\vec{\chi} - 2(\vec{\xi}\times\vec{\chi})\cdot(\vec{\phi}\times D_r\vec{\xi})\right]\,, \tag{A.6}$$

which does not receive a contribution from the triplet $\vec{\psi}$. The second HCS term, $\omega_{\mathrm{CS}}^{(2)}$, Eq. (3), however does receive a contribution from $\vec{\psi}$. The resulting expression being too cumbersome and not instructive, we do not exhibit it here. We have of course verified that its computation using symbolic manipulations is correct.

The reduced one dimensional YM Lagrangian is

$$-L_{\mathrm{YM}}^{(1)} = \left(2\,|D_r\vec{\xi}|^2 + \frac{1}{r^2}\left(|\vec{\xi}|^2 - 1\right)^2\right) - \left(r^2\,|D_r\vec{\chi}|^2 + 2\,|(\vec{\xi}\times\vec{\chi})|^2\right)\,, \tag{A.7}$$

the reduced one dimensional Higgs Lagrangian is

$$\begin{aligned}
L_{\mathrm{Higgs}} &= 2\,\eta^2\,r^2 \left\{ |(\vec{\phi}\times\vec{\chi})|^2 - \left[|D_r\vec{\phi}|^2 + \frac{2}{r^2}(\vec{\xi}\cdot\vec{\phi})^2\right] \right. \\
&\qquad \left. + |(\vec{\psi}\times\vec{\chi})|^2 - \left[|D_r\vec{\psi}|^2 + \frac{2}{r^2}(\vec{\xi}\times\vec{\psi})^2\right] \right\}\,,
\end{aligned} \tag{A.8}$$

and, finally, the Higgs potentials, Eqs. (7) and (8), reduce (for $a_1 = 1/4$ and $a_2 = 1$) to

$$v_1 = \eta^4\,r^2\left[1 - \left(|\vec{\phi}|^2 + |\vec{\psi}|^2\right)\right]^2\,, \tag{A.9}$$

$$v_2 = \eta^4\,r^2\left(\left[1 - \left(|\vec{\phi}|^2 + |\vec{\psi}|^2\right)\right]^2 + 4(\vec{\phi}\cdot\vec{\psi})^2\right)\,, \tag{A.10}$$

with

$$v_i \overset{\text{def.}}{=} r^2 V_i\,, \quad i = 1, 2\,.$$

It is clear that in the case Eq. (A.10) the asymptotic triplet $\vec{\phi}$ must be orthogonal to the asymptotic triplet $\vec{\psi}$.

Another quantity we will employ to analyze the solutions is their energy, E, given by

$$\begin{aligned}
E &= \int_0^\infty \left\{ |D_r\vec{\xi}|^2 + \frac{1}{2r^2}\left(|\vec{\xi}|^2 - 1\right)^2 + \frac{1}{2}r^2\,|D_r\vec{\chi}|^2 + |(\vec{\xi}\times\vec{\chi})|^2 \right. \\
&\quad + 2\,\eta^2\,r^2\left[|(\vec{\phi}\times\vec{\chi})|^2 + |D_r\vec{\phi}|^2 + \frac{2}{r^2}(\vec{\xi}\cdot\vec{\phi})^2 + |(\vec{\psi}\times\vec{\chi})|^2 + |D_r\vec{\psi}|^2 + \frac{2}{r^2}(\vec{\xi}\times\vec{\psi})^2\right] \\
&\quad \left. + \frac{\lambda}{2}\eta^4\,r^2\left[1 - \left(|\vec{\phi}|^2 + |\vec{\psi}|^2\right)\right]^2 \right\}\,dr\,.
\end{aligned} \tag{A.11}$$

Notice that only the first potential Eq. (7) has been included.

A.2 $SU(3)$ model

Subject to the Ansatz Eq. (50), the symmetry breaking potentials, Eqs. (7) and (8), reduce, respectively, to

$$V_1 = \eta^4\left[1 - \left(h^2 + g^2\right)\right]^2\,, \tag{A.12}$$

$$V_2 = \frac{1}{4}\eta^4\left[3 - a_2\left(h^2 + g^2\right) + \frac{a_2^2}{8}(h^2 + g^2)^2\right]\,, \tag{A.13}$$

with $a_1 = 2$. It is clear that in the case Eq. (A.13), V_2 cannot vanish for any real value of the constant a_2, i.e. we have only one choice in this case, namely Eq. (A.12).

The resulting curvatures and covariant derivative following from Eqs. (49), (50) and (51), are

$$F_{ij} = -\frac{1}{r^2}(1 - w^2)\,\lambda_{ij}^{(1)} - \left[\frac{w'}{r} + \frac{1}{r^2}(1 - w^2)\right]\hat{x}_{[i}\,\lambda_{j]k}^{(1)}\,\hat{x}_k\,, \tag{A.14}$$

$$D_i\Phi = \frac{1}{2}\,i\,\eta\left[\frac{wh}{r}\,\lambda_i^{(1)} + \left(h' - \frac{wh}{r}\right)\hat{x}_i\hat{x}_j\lambda_j^{(1)} + g'\,\hat{x}_i\,\lambda_8\right]\,, \tag{A.15}$$

$$F_{i0} = \frac{1}{2}\,i\left[\frac{wu}{r}\,\lambda_i^{(1)} + \left(u' - \frac{wu}{r}\right)\hat{x}_i\hat{x}_j\lambda_j^{(1)} + v'\,\hat{x}_i\,\lambda_8\right]\,, \tag{A.16}$$

$$D_0\Phi = 0\,, \tag{A.17}$$

further resulting in

$$\mathrm{Tr}\, F_{ij}^2 \;=\; -\frac{1}{r^2}\left[2\,w'^2 + \frac{1}{r^2}(1-w^2)^2\right]\,, \tag{A.18}$$

$$\mathrm{Tr}\, F_{i0}^2 \;=\; -\frac{1}{2}\left[u'^2 + \frac{2}{r^2}w^2u^2 + v'^2\right]\,, \tag{A.19}$$

$$\mathrm{Tr}\, D_i\Phi^2 \;=\; -\frac{1}{2}\eta^2\left[h'^2 + \frac{2}{r^2}w^2h^2 + g'^2\right]\,. \tag{A.20}$$

The magnetic charge integral, Eq. (9), reduces to

$$\mu = \eta[(1-w^2)h]_{r=\infty}\,, \tag{A.21}$$

and the electric charge integral, Eq. (10), results to be

$$Q = \frac{1}{2}\eta\left[r^2\,(hu' + gv')\right]_{r=\infty}\,. \tag{A.22}$$

The energy of the solutions is given by

$$E = \frac{1}{4}\int_0^\infty \left[r^2u'^2 + r^2v'^2 + 2u^2w^2 + \frac{(1-w^2)^2}{r^2} + 2w'^2 + r^2g'^2 + r^2h'^2 + 2h^2w^2 + 2\lambda r^2(1-g^2-h^2)^2\right]\,dr\,. \tag{A.23}$$

Subject to this spherical symmetry, the HCS densities Eqs. (2) and (3), do not identically vanish but yield

$$\Omega_{\mathrm{CS}}^{(1)} = -\frac{2}{\sqrt{3}r^2}\eta\left[(1-w^2)\,h\,v' + g[(1-w^2)u]'\right]\,, \tag{A.24}$$

$$\Omega_{\mathrm{CS}}^{(2)} = -\frac{\sqrt{3}}{54r^2}\eta^3\left\{-2guw(36-g^2-5h^2)w' + 3h\left[(1-w^2)(12-g^2-h^2)+2h^2w^2\right]v' \right.$$
$$\left. +g\left[(1-w^2)(36-g^2-9h^2)+2h^2w^2\right]u' - h^2uw^2g' + 4ghuw^2h'\right\}\,. \tag{A.25}$$

B   Imposition of axial symmetry on the $SU(3)$ model

In this Appendix, we present the axially symmetric field configurations employed in Section **4.3**, in the discussion of the issue of angular momentum in the $SU(3)$ model.

We denote the magnetic component $A_i = (A_\alpha, A_z)$ of the $SU(3)$ connection corresponding to the spherically symmetric Ansatz Eq. (49) as

$$A_i = \begin{bmatrix} \hat{A}_i & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 2} \end{bmatrix}\,; \quad i = \alpha, z \equiv \alpha, 3\,; \quad \alpha = x, y \equiv 1, 2\,, \tag{B.1}$$

The electric component A_0 of the $SU(3)$ connection corresponding to the spherically symmetric Ansatz Eq. (51) and the Higgs field Φ corresponding to Eq. (50), likewise

$$A_0 = \begin{bmatrix} \hat{A}_0 & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 2} \end{bmatrix} + i\,v(\rho,z)\,\lambda_8\,, \tag{B.2}$$

and

$$(2\eta)^{-2}\,\Phi = \begin{bmatrix} \hat{\Phi} & 0_{1\times 2} \\ 0_{2\times 1} & 0_{2\times 2} \end{bmatrix} + i\,g(\rho,z)\,\lambda_8\,, \tag{B.3}$$

respectively.

There now remains to impose axial symmetry on the $SU(2)$ algebra valued quantities $\hat{A}_i = (\hat{A}_\alpha, \hat{A}_z)$, $\hat{A}_0$ and $\hat{\Phi}$. For this, we employ the chiral $SO(4)$ matrices [5] $\Sigma_{MN}^{(\pm)}$ representing the $SU_{(\pm)}(2)$ subalgebra valued quantities in Eqs. (B.1)-(B.3)

In this notation,

$$\hat{A}_\alpha = \left(\frac{\xi^2 + n}{\rho}\right)(\varepsilon\hat{x})_\alpha \Sigma_{12} + \left[\left(\frac{\xi^1}{\rho}\right)(\varepsilon\hat{x})_\alpha (\varepsilon n)_\gamma + a_\rho \hat{x}_\alpha n_\gamma\right]\Sigma_{\gamma 3}, \tag{B.8}$$

$$\hat{A}_z = a_z n_\gamma \Sigma_{\gamma 3}, \tag{B.9}$$

$$\hat{A}_0 = -\chi^1 n_\gamma \Sigma_{\gamma 4} + \chi^2 \Sigma_{34} = \chi^1 (\varepsilon n)_\gamma \Sigma_{\gamma 3} + \chi^2 \Sigma_{12}, \tag{B.10}$$

$$\hat{\Phi} = -\phi^1 n_\gamma \Sigma_{\gamma 4} + \phi^2 \Sigma_{34} = \phi^1 (\varepsilon n)_\gamma \Sigma_{\gamma 3} + \phi^2 \Sigma_{12}, \tag{B.11}$$

where $n_\alpha = (\cos n\varphi, \sin n\varphi)$ is the unit vector in the (x_1, x_2) plane, φ is the azimuthal angle and n is the vortex number. The functions (a_ρ, a_z), $\xi^a = (\xi^1, \xi^2)$, $\chi^a = (\chi^1, \chi^2)$ and $\phi^a = (\phi^1, \phi^2)$ all depend on the two variables $\rho = \sqrt{|x_\alpha|^2}$ and z, and are independant of the time coordinate x_0.

The gauge covariant quantities $F_{\mu\nu} = (F_{\alpha\beta}, F_{\alpha z}, F_{\alpha 0}, F_{z0})$ and $D_\mu \Phi = (D_\alpha \Phi, D_z \Phi, D_0 \Phi)$ follow,

$$\hat{F}_{\alpha\beta} = -\frac{1}{\rho}\varepsilon_{\alpha\beta}\left[D_\rho \xi^1 (\varepsilon n)_\gamma \Sigma_{\gamma 3} + D_\rho \xi^2 \Sigma_{12}\right], \tag{B.12}$$

$$\hat{F}_{\alpha z} = f_{\rho z} x_\alpha n_\gamma \Sigma_{\gamma 3} - \frac{1}{\rho}(\varepsilon\hat{x})_\alpha \left[D_z \xi^1 (\varepsilon n)_\gamma \Sigma_{\gamma 3} + D_z \xi^2 \Sigma_{12}\right], \tag{B.13}$$

$$\hat{F}_{\alpha 0} = \frac{1}{\rho}(\chi\varepsilon\xi)(\varepsilon\hat{x})_\alpha n_\gamma \Sigma_{\gamma 3} + \hat{x}_\alpha \left[D_\rho \chi^1 (\varepsilon n)_\gamma \Sigma_{\gamma 3} + D_\rho \chi^2 \Sigma_{12}\right], \tag{B.14}$$

$$\hat{F}_{z0} = D_z \chi^1 (\varepsilon n)_\gamma \Sigma_{\gamma 3} + D_z \chi^2 \Sigma_{12}, \tag{B.15}$$

and

$$D_\alpha \hat{\Phi} = \hat{x}_\alpha \left[D_\rho \phi^1 (\varepsilon n)_\gamma \Sigma_{\gamma 3} + D_\rho \phi^2 \Sigma_{12}\right] + \frac{1}{\rho}(\phi\varepsilon\xi)(\varepsilon\hat{x})_\alpha n_\gamma \Sigma_{\gamma 3}, \tag{B.16}$$

$$D_z \hat{\Phi} = D_z \phi^1 (\varepsilon n)_\gamma \Sigma_{\gamma 3} + D_z \phi^2 \Sigma_{12}, \tag{B.17}$$

$$D_0 \hat{\Phi} = (\phi\varepsilon\chi) n_\gamma \Sigma_{\gamma 3}, \tag{B.18}$$

which are all expressed in terms of the $SO(2)$ curvature

$$f_{\rho z} = \partial_\rho a_z - \partial_z a_\rho,$$

the $SO(2)$ covariant derivatives

$$D_\rho \xi^a = \partial_\rho \xi^a + a_\rho (\varepsilon\xi)^a, \quad D_z \xi^a = \partial_z \xi^a + a_z (\varepsilon\xi)^a, \quad etc.$$

and with

$$(f\varepsilon g) = \varepsilon^{ab} f^a g^b.$$

---

[5]

$$\Sigma_{MN}^{(+)} = -\frac{1}{4}\left(\Sigma_M \tilde{\Sigma}_N - \Sigma_N \tilde{\Sigma}_M\right), \tag{B.4}$$

$$\Sigma_{MN}^{(-)} = -\frac{1}{4}\left(\tilde{\Sigma}_M \Sigma_N - \tilde{\Sigma}_N \Sigma_M\right), \tag{B.5}$$

where the index $M = \alpha, 3, 4$, with $\alpha = 1, 2$. The spin matrices used are

$$\Sigma_\alpha = -\tilde{\Sigma}_\alpha = i\sigma_\alpha, \quad \Sigma_3 = -\tilde{\Sigma}_3 = i\sigma_3, \quad \Sigma_4 = \tilde{\Sigma}_4 = \mathbb{1}, \tag{B.6}$$

where $(\sigma_\alpha, \sigma_3)$ are the usual 2×2 Pauli spin matrices.

The matrices $\Sigma_{MN}^{(\pm)}$ are (anti)self-dual

$$\Sigma_{MN}^{(\pm)} = \pm\frac{1}{2}\varepsilon_{MNRS}\Sigma_{RS}^{(\pm)}, \tag{B.7}$$

In particular, we opt for the selfdual case.

The (static) axially symmetric $U(1) \simeq SO(2)$ gauge connection $a_\mu = (a_\alpha, a_z, a_0)$ can be expressed as

$$a_\alpha = u(r,\theta) \, (\hat{x}\varepsilon)_\alpha \,, \tag{B.19}$$

$$a_z = 0 \,, \tag{B.20}$$

$$a_0 = a_0(r,\theta) \,. \tag{B.21}$$

In the calcualtion of the angular momentum, the azimuthal component of the Abelian connection a_φ will be employed, which in the notation of Eq. (B.19) is

$$a_\varphi = \rho \, u \,. \tag{B.22}$$

The components of the Abelian curvature $h_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ follow

$$h_{\alpha\beta} = \frac{1}{\rho} \, (\rho \, u)_{,\rho} \, \varepsilon_{\alpha\beta} \,, \tag{B.23}$$

$$h_{\alpha z} = u_{,z} \, (\varepsilon\hat{x})_\alpha \,, \tag{B.24}$$

$$h_{\alpha 0} = (a_0)_{,\rho} \, \hat{x}_\alpha \,, \tag{B.25}$$

$$h_{z0} = (a_0)_{,z} \,. \tag{B.26}$$

The reduced two dimensional Lagrangian is

$$
\begin{aligned}
L^{(1)} = & -\frac{1}{4} \left\{ \frac{1}{\rho} \left(|D_\rho\xi|^2 + |D_z\xi|^2 \right) - \rho \left(|D_\rho\chi|^2 + |D_z\chi|^2 \right) + \rho \, f_{\rho z}^2 - \frac{1}{\rho}(\chi\varepsilon\xi)^2 - 4\rho \left[\partial_\rho v^2 + \partial_z v^2 \right] \right\} \\
& -\eta^2 \left\{ \rho \left(|D_\rho\phi|^2 + |D_z\phi|^2 \right) + \frac{1}{\rho}(\phi\varepsilon\xi)^2 - \rho(\phi\varepsilon\chi)^2 - 4\rho \left[\partial_\rho g^2 + \partial_z g^2 \right] \right\} + \kappa_1 \, \omega^{(1)} \,,
\end{aligned} \tag{B.27}
$$

where $\omega^{(1)}$ is the reduced two dimensional HCS density Eq. (2),

$$\omega^{(1)} = \frac{8}{\sqrt{3}} \, \eta \left\{ g \left[(\chi\varepsilon\xi) \, f_{\rho z} - D_{[\rho}\xi \cdot D_{z]}\chi \right] + \partial_{[\rho}v(\phi \cdot D_{z]}\xi) \right\} \,. \tag{B.28}$$

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