A note on operator tuples which are (m, p)-isometric as well as (μ, ∞)-isometric

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Abstract

We show that if a tuple of commuting, bounded linear operators $(T_1, ..., T_d) \in B(X)^d$ is both an (m,p)-isometry and a (μ, ∞)-isometry, then the tuple $(T_1^m, ..., T_d^m)$ is a $(1,p)$-isometry. We further prove some additional properties of the operators $T_1, ..., T_d$ and show a stronger result in the case of a commuting pair (T_1, T_2).

Keywords: operator tuple, normed space, Banach space, m-isometry, (m,p)-isometry, (m,∞)-isometry

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1 Introduction

Let in the following X be a normed vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let the symbol $\mathbb{N}$ denote the natural numbers including 0.

A tuple of commuting linear operators $T := (T_1, ..., T_d)$ with $T_j : X \to X$ is called an (m,p)-*isometry* (or an (m,p)-*isometric tuple*) if, and only if, for given $m \in \mathbb{N}$ and $p \in (0, \infty)$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^\alpha x\|^p = 0, \quad \forall x \in X. \tag{1.1}$$

Here, $\alpha := (\alpha_1, ..., \alpha_d) \in \mathbb{N}^d$ is a multi-index, $|\alpha| := \alpha_1 + \cdots + \alpha_d$ the sum of its entries, $\frac{k!}{\alpha!} := \frac{k!}{\alpha_1! \cdots \alpha_d!}$ a multinomial coefficient and $T^\alpha := T_1^{\alpha_1} \cdots T_d^{\alpha_d}$, where $T_j^0 := I$ is the identity operator.

Tuples of this kind have been introduced by Gleason and Richter [10] on Hilbert spaces (for $p = 2$) and have been further studied on general normed spaces in [8]. The tuple case generalises the single operator case, originating in the works of Richter [11] and Agler [2] in the 1980s and being comprehensively studied in the Hilbert space case by Agler and Stankes [3]; the single operator case on Banach spaces has been introduced by Bayart in [4] in its general form and also has also been studied in [7] and [12]. We remark that boundedness, although usually assumed, is not essential for the definition of (m,p)-isometries, as shown by Bermúdez, Martinón and Müller in [5]. Boundedness does, however, play an important role in the theory of objects of the following kind:

Let $B(X)$ denote the algebra of bounded (i.e. continuous) linear operators on X. Equating sums over even and odd k and then considering $p \to \infty$ in

(1.1), leads to the definition of (m,∞)-*isometries* (or (m,∞)-*isometric tuples*). That is, a tuple of commuting, bounded linear operators $T \in B(X)^d$ is referred to as an (m,∞)-isometry if, and only if, for given $m \in \mathbb{N}$ with $m \geq 1$,

$$\max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ even}}} \|T^\alpha x\| = \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ odd}}} \|T^\alpha x\|, \quad \forall x \in X. \tag{1.2}$$

These tupes have been introduced in [8], with the definition of the single operator case appearing in [9]. Although, it may be possible that tuples of unbounded operators satisfying (1.2) exist, several important statements on (m,∞)-isometries require boundedness. Therefore, from now on, we will always assume the operators $T_1,\dots,T_d$ to be bounded.

In [8], the question is asked what necessary properties a commuting tuple $T \in B(X)^d$ has to satisfy if it is both an (m,p)-isometry and a (μ,∞)-isometry, where possibly $m \neq \mu$. In the single operator case this question is trivial and answered in [9]: If $T = T_1$ is a single operator, then the condition that T_1 is an (m,p)-isometry is equivalent to the mapping $n \mapsto \|T_1^n x\|^p$ being a polynomial of degree $\leq m-1$ for all $x \in X$. This has been already been observed for operators on Hilbert spaces in [10] and shown in the Banach space/normed space case in [9]; the necessity of the mapping $n \mapsto \|T_1^n x\|^p$ being a polynomial has already been proven in [4] and [6]. On the other hand, in [9] it is shown that if a bounded operator $T = T_1 \in B(X)$ is a (μ,∞)-isometry, then the mapping $n \mapsto \|T_1^n x\|$ is bounded for all $x \in X$. The conclusion is obvious: if $T = T_1 \in B(X)$ is both (m,p)- and (μ,∞)-isometric, then $n \mapsto \|T_1^n x\|^p$ is always constant and T_1 has to be an isometry (and, since every isometry is $(m,p)-$ and (μ,∞)-isometric, we have equivalence).

The situation is, however, far more difficult in the multivariate, that is, in the operator tuple case. Again, we have equivalence between $T = (T_1,\dots,T_d)$ being an (m,p)-isometry and the mapping $n \mapsto \sum_{|\alpha|=n} \frac{n!}{\alpha!}\|T^\alpha x\|^p$ being polynomial of degree $\leq m-1$ for all $x \in X$. The necessity part of this statement has been proven in the Hilbert space case in [10] and equivalence in the general case has been shown in [8]. On the other hand, one can show that if $T \in B(X)^d$ is a (μ,∞)-isometry, then the family $(\|T^\alpha x\|)_{\alpha\in\mathbb{N}^d}$ is bounded for all $x \in X$, which has been proven in [8]. But this fact only implies that the polynomial growth of $n \mapsto \sum_{|\alpha|=n} \frac{n!}{\alpha!}\|T^\alpha x\|^p$ has to caused by the factors $\frac{n!}{\alpha!}$ and does not immediately give us any further information about the tuple T.

There are several results in special cases proved in [8]. For instance, if a commuting tuple $T = (T_1,\dots,T_d) \in B(X)^d$ is an (m,p)-isometry as well as a (μ,∞)-isometry and we have $m = 1$ or $\mu = 1$ or $m = \mu = d = 2$, then there exists one operator $T_{j_0} \in \{T_1,\dots,T_d\}$ which is an isometry and the remaining operators T_k for $k \neq j_0$ are in particular nilpotent of order m. Although, we are not able to obtain such a results for general $m \in \mathbb{N}$ and $\mu, d \in \mathbb{N}\setminus\{0\}$, yet, we can prove a weaker property: In all proofs of the cases discussed in [8], the fact that the tuple $(T_1^m,\dots,T_d^m)$ is a $(1,p)$-isometry is of critical importance (see the proofs of [8, Theorem 7.1 and Proposition 7.3]). We will show in this paper that this fact holds in general for any tuple which is both (m,p)-isometric and (μ,∞)-isometric, for general m, μ and d.

The notation we will be using is basically standard, with one possible exception: We will denote the tuple of $d-1$ operators obtained by removing one operator T_{j_0} from $(T_1,\dots,T_d)$ by T'_{j_0}, that is $T'_{j_0} := (T_1,\dots,T_{j_0-1},T_{j_0+1},\dots,T_d) \in$

$B(X)^{d-1}$ (not to be confused with the dual of the operator T_{j_0}, which will not appear in this paper). Analogously, we denote by α'_{j_0} the multi-index obtained by removing α_{j_0} from $(\alpha_1, ..., \alpha_d)$.

We will further use the notations $R(T_j)$ for the range and $N(T_j)$ for the kernel (or nullspace) of an operator T_j.

2 Preliminaries

In this section, we introduce two needed definitions/notations and compile a number of propositions and theorems, predominantly taken from [8], which are necessary for our considerations.

In the following, for $T \in B(X)^d$ and given $p \in (0,\infty)$, define for all $x \in X$ the sequences $(Q^{n,p}(T,x))_{n\in\mathbb{N}}$ by

$$Q^{n,p}(T,x) := \sum_{|\alpha|=n} \frac{n!}{\alpha!}\|T^\alpha x\|^p.$$

Define further for all $\ell \in \mathbb{N}$ and all $x \in X$, the mappings $P_\ell^{(p)}(T,\cdot) : X \to \mathbb{R}$, by

$$\begin{aligned} P_\ell^{(p)}(T,x) :=& \sum_{k=0}^{\ell}(-1)^{\ell-k}\binom{\ell}{k}Q^{k,p}(T,x) \\ =& \sum_{k=0}^{\ell}(-1)^{\ell-k}\binom{\ell}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha x\|^p. \end{aligned}$$

It is clear that $T \in B(X)^d$ is an (m,p)-isometry if, and only if, $P_m^{(p)}(T,\cdot) \equiv 0$.

If the context is clear, we will simply write $P_\ell(x)$ and $Q^n(x)$ instead of $P_\ell^{(p)}(T,x)$ and $Q^{n,p}(T,x)$.

Further, for $n,k \in \mathbb{N}$, define the (descending) Pochhammer symbol $n^{(k)}$ as follows:

$$n^{(k)} := \begin{cases} 0, & \text{if } k > n, \\ \binom{n}{k}k!\ , & \text{else.} \end{cases}$$

Then $n^{(0)} = 0^{(0)} = 1$ and, if $n,k > 0$ and $k \leq n$, we have

$$n^{(k)} = n(n-1)\cdots(n-k+1).$$

As mentioned above, a fundamental property of (m,p)-isometries is that their defining property can be expressed in terms of polynomial sequences.

Theorem 2.1 ([8, Theorem 3.1])**.** *$T \in B(X)^d$ is an (m,p)-isometry if, and only if, there exists a family of polynomials $f_x : \mathbb{R} \to \mathbb{R}$, $x \in X$, of degree $\leq m-1$ with $f_x|_{\mathbb{N}} = (Q^n(x))_{n\in\mathbb{N}}$.*[1]

This actually follows by the (not immediate[2]) application of a well-known theorem about functions defined on the natural numbers, which itself will be needed for our considerations as well. We give it here in a simplified form which is sufficient for our needs.

[1] Set $\deg 0 := -\infty$ to account for the case $m = 0$.

[2] The application of Theorem 2.2 to (m,p)-isometries by setting $a = (Q_n(x))_{n\in\mathbb{N}}$ is not immediate, since the requirement $P_m(T,x) = 0$ is only the case $n = 0$ in (2.1).

Theorem 2.2 (see, for instance, [1, Satz 3.1]). *Let $a = (a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence and $m \in \mathbb{N}$. Then we have*

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{n+k} = 0, \quad \forall n \in \mathbb{N} \tag{2.1}$$

if, and only if, there exists a polynomial function f of degree $\deg f \leq m-1$ with $f|_{\mathbb{N}} = a$.[1]

Two important consequences of Theorem 2.1 are contained in the following corollary. The first part describes the Newton-form of the Lagrange-polynomial f_x interpolating $(Q^n(x))_{n\in\mathbb{N}}$. The second part trivially describes the leading coefficient of f_x.

Corollary 2.3 ([8, Proposition 3.2]). *Let $m \geq 1$ and $T \in B(X)^d$ be an (m,p)-isometry. Then we have*

(i) for all $n \in \mathbb{N}$

$$Q^n(x) = \sum_{k=0}^{m-1} n^{(k)} \left(\frac{1}{k!} P_k(x) \right), \quad \forall x \in X;$$

(ii)

$$\lim_{n\to\infty} \frac{Q^n(x)}{n^{m-1}} = \frac{1}{(m-1)!} P_{m-1}(x) \geq 0, \quad \forall x \in X.$$

Regarding (m,∞)-isometries, we will need the following two statements. Theorem 2.5 is a combination of several fundamental properties of (m,∞)-isometric tuples.

Proposition 2.4 ([8, Corollary 5.1]). *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an (m,∞)-isometry. Then $(\|T^\alpha x\|)_{\alpha\in\mathbb{N}^d}$ is bounded, for all $x \in X$, and*

$$\max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\| = \max_{|\alpha|=0,\dots,m-1} \|T^\alpha x\|,$$

for all $x \in X$.

Theorem 2.5 ([8, Proposition 5.5, Theorem 5.1 and Remark 5.2]). *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an (m,∞)-isometric tuple. Define the norm $|.|_\infty : X \to [0,\infty)$ via $|x|_\infty := \max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\|$, for all $x \in X$, and denote*

$$X_{j,|.|_\infty} := \{x \in X \mid |x|_\infty = |T_j^n x|_\infty \text{ for all } n \in \mathbb{N}\}.$$

Then

$$X = \bigcup_{j=1,\dots,d} X_{j,|.|_\infty}.$$

(Note that, by Proposition 2.4, $|.|_\infty = \|.\|$ if $m = 1$.)

We will also require a fundamental fact on tuples which are both (m,p)- and (μ,∞)-isometric and an (almost) immediate corollary.

Lemma 2.6 ([8, Lemma 7.2]). *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an (m,p)-isometry as well as a (μ,∞)-isometry. Let $\gamma = (\gamma_1, ..., \gamma_d) \in \mathbb{N}^d$ be a multi-index with the property that $|\gamma_j'| \geq m$ for every $j \in \{1, ..., d\}$. Then $T^\gamma = 0$.*

Conversely, this implies that if an operator T^α is not the zero-operator, the multi-index α has to be of a specific form. The proof in [8] of the following corollary appears to be overly complicated, the statement is just the negation of the previous lemma.

Corollary 2.7 ([8, Corollary 7.1]). *Let $T=(T_1,...,T_d)\in B(X)^d$ be an (m,p)-isometry for some $m\geq 1$ as well as a (μ,∞)-isometry. If $\alpha\in\mathbb{N}^d$ is a multi-index with $T^\alpha\neq 0$ and $|\alpha|=n$, then there exists some $j_0\in\{1,...,d\}$ with $T^\alpha=T_{j_0}^{n-|\alpha'_{j_0}|}(T'_{j_0})^{\alpha'_{j_0}}$ and $|\alpha'_{j_0}|\leq m-1$.*

This fact has consequences for the appearance of elements of the sequences $(Q^n(x))_{n\in\mathbb{N}}$, since several summands become zero for large enough n. That is, we have trivially by definition of $(Q^n(x))_{n\in\mathbb{N}}$:

Corollary 2.8 ([8, proof of Theorem 7.1]). *Let $T=(T_1,...,T_d)\in B(X)^d$ be an (m,p)-isometry for some $m\geq 1$ as well as a (μ,∞)-isometry. Then, for all $n\in\mathbb{N}$ with $n\geq 2m-1$, we have*

$$Q^n(x)=\sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=0,...,m-1}}\sum_{j=1}^{d}\frac{n!}{(n-|\beta|)!\beta!}\|T_j^{n-|\beta|}(T'_j)^\beta x\|^p,\quad\forall x\in X,$$

where $\frac{n!}{(n-|\beta|)!\beta!}=\frac{n^{(|\beta|)}}{\beta!}$. (We set $n\geq 2m-1$ to ensure that every multi-index only appears once.)

3 The main result

We first present the main result of this article, which is a generalisation of [8, Proposition 7.3], before stating a preliminary lemma needed for its proof.

Theorem 3.1. *Let $T=(T_1,...,T_d)\in B(X)^d$ be an (m,p)-isometric as well a (μ,∞)-isometric tuple. Then*

(i) the sequences $n\mapsto\|T_j^n x\|$ become constant for $n\geq m$, for all $j\in\{1,...,d\}$, for all $x\in X$.

(ii) the tuple $(T_1^m,...,T_d^m)$ is a $(1,p)$-isometry, that is

$$\sum_{j=1}^{d}\|T_j^m x\|^p=\|x\|^p,\quad\forall x\in X.$$

(iii) for any $(n_1,...,n_d)\in\mathbb{N}^d$ with $n_j\geq m$ for all j, the operators $\sum_{j=1}^d T_j^{n_j}$ are isometries, that is

$$\left\|\sum_{j=1}^{d}T_j^{n_j}x\right\|=\|x\|,\quad\forall x\in X.$$

Of course, (i) and (ii) imply that, for any $(n_1,...,n_d)\in\mathbb{N}^d$ with $n_j\geq m$ for all j,

$$\sum_{j=1}^{d}\|T_j^{n_j}x\|^p=\|x\|^p,\quad \forall x\in X,$$

Theorem 3.1 is a consequence of the following lemma, which is a weaker version of 3.1.(i).

Lemma 3.2. *Let $T=(T_1,...,T_d)\in B(X)^d$ be an (m,p)-isometric as well as a (μ,∞)-isometric tuple. Let further $\kappa\in\mathbb{N}^{d-1}$ be a multi-index with $|\kappa|\geq 1$. Then the mappings*

$$n\mapsto\|T_j^n\left(T_j'\right)^{\kappa}x\|$$

become constant for $n\geq m$, for all $j\in\{1,...,d\}$, for all $x\in X$.

Proof. If $m=0$, then $X=\{0\}$ and if $m=1$, the statement holds trivially, since $T_jT_i=0$ for all $i\neq j$ by Lemma 2.6. So assume $m\geq 2$. Further, it clearly suffices to consider $|\kappa|=1$, since the statement then holds for all $x\in X$. The proof, however, works by proving the theorem for $|\kappa|\in\{1,...,m-1\}$ in descending order. (Note that the case $|\kappa|\geq m$ is also trivial, again by Lemma 2.6.)

Since for $n\geq 2m-1$, by Corollary 2.8,

$$Q^n(x)=\sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=0,...,m-1}}\sum_{j=1}^{d}\frac{n^{(|\beta|)}}{\beta!}\|T_j^{n-|\beta|}(T_j')^{\beta}x\|^p,\quad\forall x\in X,$$

and $P_{m-1}(x)=\lim_{n\to\infty}\frac{Q^n(x)}{n^{m-1}}$, for all $x\in X$, by Corollary 2.3.(ii), we have that

$$P_{m-1}(x)=\lim_{n\to\infty}\sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=m-1}}\sum_{j=1}^{d}\frac{1}{\beta!}\|T_j^{n}(T_j')^{\beta}x\|^p,\quad\forall x\in X.$$

Now fix an arbitrary $j_0\in\{1,...,d\}$ and let $\kappa\in\mathbb{N}^{d-1}$ with $|\kappa|\in\{1,...,m-1\}$. Again, by Lemma 2.6, we have, for any $\nu\geq 1$,

$$P_{m-1}\left(T_{j_0}^{\nu}\left(T_{j_0}'\right)^{\kappa}x\right)=0,\quad\forall x\in X. \tag{3.1}$$

Now let $\nu\geq m$ and set $\ell:=m-|\kappa|$. Then $\ell\in\{1,...,m-1\}$ and $|\kappa|=m-\ell$.

We again apply Lemma 2.6, this time to $Q^k(T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\kappa} x)$. By definition,

$$
\begin{aligned}
&Q^k(T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\kappa} x) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \|T^{\alpha}\left(T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\kappa} x\right)\|^p \\
&= \|T_{j_0}^{k}\left(T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\kappa} x\right)\|^p + \sum_{j=1}^{k} \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\ |\beta|=j}} \frac{k!}{(k-j)!\beta!} \|T_{j_0}^{k-j}\left(T'_{j_0}\right)^{\beta}\left(T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\kappa} x\right)\|^p \\
&\overset{2.6}{=} \|T_{j_0}^{\nu+k}\left(T'_{j_0}\right)^{\kappa} x\|^p + \sum_{j=1}^{\min\{k,\ell-1\}} \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\ |\beta|=j}} \frac{k!}{(k-j)!\beta!} \|T_{j_0}^{\nu+k-j}\left(T'_{j_0}\right)^{\kappa+\beta} x\|^p \\
&= \|T_{j_0}^{\nu+k}\left(T'_{j_0}\right)^{\kappa} x\|^p + \sum_{j=1}^{\ell-1} k^{(j)} \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\ |\beta|=j}} \frac{1}{\beta!} \|T_{j_0}^{\nu+k-j}\left(T'_{j_0}\right)^{\kappa+\beta} x\|^p,
\end{aligned}
$$

for all $k\in\mathbb{N}$, for all $x\in X$. Here, in the third line, the fact that $\nu\geq m$ is used, where in the last line, we utilise the fact that $k^{(j)}=0$ if $j>k$.

We now prove our statement by (finite) induction on ℓ.

<u>$\ell = 1$:</u>
For $\ell=1$ and $|\kappa|=m-1$, we have

$$Q^k\left(T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\kappa} x\right) = \|T_{j_0}^{\nu+k}\left(T'_{j_0}\right)^{\kappa} x\|^p, \quad \forall k\in\mathbb{N},\ \forall x\in X.$$

Hence, since $P_{m-1}(x) = \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}Q^k(x)$ by definition, we have, by (3.1),

$$P_{m-1}\left(T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\kappa} x\right) = \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\|T_{j_0}^{\nu+k}\left(T'_{j_0}\right)^{\kappa} x\|^p = 0, \quad \forall x\in X.$$

However, by definition, that means, that the operator $T_{j_0}|_{R(T_{j_0}^{\nu}(T'_{j_0})^{\kappa})}$ (that is, T_{j_0} restricted to the range of $T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\kappa}$) is an $(m-1,p)$-isometric operator.

By Theorem 2.1 (or, as mentioned in the introduction, by statements proven by earlier authors), this implies that the sequences $n\mapsto\|T_{j_0}^{n+\nu}\left(T'_{j_0}\right)^{\kappa} x\|^p$ is polynomial of degree $\leq m-2$, for all $x\in X$. Thus, $n\mapsto\|T_{j_0}^{n}\left(T'_{j_0}\right)^{\kappa} x\|^p$, become polynomial of degree $\leq m-2$, for $n\geq\nu\geq m$, for all $x\in X$.

However, since T is a (μ,∞)-isometric tuple, by Proposition 2.4 the sequences $n\mapsto\|T_j^n x\|$ are bounded for all $j\in\{1,...,d\}$, for all $x\in X$. Therefore, we must have that the mappings

$$n\mapsto\|T_{j_0}^{n}\left(T'_{j_0}\right)^{\kappa} x\|$$

become constant for $n\geq m$, for all $x\in X$.

Since $\ell\in\{1,...,m-1\}$, if we had $m=2$, we are already done. So assume in the following that $m\geq 3$.

<u>$\ell \to \ell+1$:</u>

Assume that the statement holds for some $\ell \in \{1,...,m-2\}$. That is, for all $\kappa \in \mathbb{N}^{d-1}$ with $|\kappa| = m-\ell$ the sequences

$$n \mapsto \|T_{j_0}^n \left(T_{j_0}'\right)^\kappa x\|$$

become constant for $n \geq m$, for all $x \in X$.

Now take a multi-index $\tilde\kappa \in \mathbb{N}^{d-1}$ with $|\tilde\kappa| = m-(\ell+1)$ and consider

$$Q^k(T_{j_0}^\nu \left(T_{j_0}'\right)^{\tilde\kappa} x) = \|T_{j_0}^{\nu+k} \left(T_{j_0}'\right)^{\tilde\kappa} x\|^p + \sum_{j=1}^{\ell-1} k^{(j)} \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=j}} \frac{1}{\beta!}\|T_{j_0}^{\nu+k-j} \left(T_{j_0}'\right)^{\tilde\kappa+\beta} x\|^p.$$

Note that we have $|\tilde\kappa+\beta| \geq m-\ell$, since $|\beta| \geq 1$. Hence, if $k \geq j$, by our induction assumption,

$$\|T_{j_0}^{\nu+k-j} \left(T_{j_0}'\right)^{\tilde\kappa+\beta} x\|^p = \|T_{j_0}^{\nu} \left(T_{j_0}'\right)^{\tilde\kappa+\beta} x\|^p, \quad \forall x \in X,$$

since $n \mapsto \|T_{j_0}^n \left(T_{j_0}'\right)^{\tilde\kappa+\beta} x\|$ become constant for $n \geq \nu \geq m$.

Hence, we have

$$Q^k(T_{j_0}^\nu \left(T_{j_0}'\right)^{\tilde\kappa} x) = \|T_{j_0}^{\nu+k} \left(T_{j_0}'\right)^{\tilde\kappa} x\|^p + \sum_{j=1}^{\ell-1} k^{(j)} \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=j}} \frac{1}{\beta!}\|T_{j_0}^{\nu} \left(T_{j_0}'\right)^{\tilde\kappa+\beta} x\|^p.$$

Then, by definition and 3.1,

$$\begin{aligned} 0 =& P_{m-1}\left(T_{j_0}^\nu \left(T_{j_0}'\right)^{\tilde\kappa} x\right) = \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}Q^k(x) \\ =& \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\|T_{j_0}^{\nu+k} \left(T_{j_0}'\right)^{\tilde\kappa} x\|^p \\ &+ \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\left(\sum_{j=1}^{\ell-1} k^{(j)} \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=j}} \frac{1}{\beta!}\|T_{j_0}^{\nu} \left(T_{j_0}'\right)^{\tilde\kappa+\beta} x\|^p\right), \end{aligned}$$

for all $x \in X$. But now, for all $x \in X$, the sequence

$$k \mapsto \left(\sum_{j=1}^{\ell-1} k^{(j)} \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=j}} \frac{1}{\beta!}\|T_{j_0}^{\nu} \left(T_{j_0}'\right)^{\tilde\kappa+\beta} x\|^p\right)$$

is polynomial (in k) of degree $\leq \ell-1 \leq m-3$ (with trailing coefficient 0). Hence, by Theorem 2.2,

$$\sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\left(\sum_{j=1}^{\ell-1} k^{(j)} \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=j}} \frac{1}{\beta!}\|T_{j_0}^{\nu} \left(T_{j_0}'\right)^{\tilde\kappa+\beta} x\|^p\right) = 0$$

and, thus,

$$0 = P_{m-1}\left(T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\tilde{\kappa}}x\right) = \sum_{k=0}^{m-1}(-1)^{m-1-k}\binom{m-1}{k}\|T_{j_0}^{\nu+k}\left(T'_{j_0}\right)^{\tilde{\kappa}}x\|^p,$$

for all $x \in X$. Now we can repeat the argument from the case $\ell = 1$ (that is, T_{j_0} restricted to the range of $T_{j_0}^{\nu}\left(T'_{j_0}\right)^{\tilde{\kappa}}$ is an $(m-1,p)$-isometric operator), to obtain again that the sequences

$$n \mapsto \|T_{j_0}^{n}\left(T'_{j_0}\right)^{\tilde{\kappa}}x\|$$

become constant for $n \geq \nu \geq m$, for all $x \in X$. This concludes the induction step and the proof. □

We can now prove the main result.

Proof of Theorem 3.1. By the lemma above, we have for $n \geq 2m-1$,

$$\begin{aligned} Q^n(x) &= \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\ |\beta|=0,\dots,m-1}} n^{(|\beta|)}\sum_{j=1}^{d}\frac{1}{\beta!}\|T_j^{n-|\beta|}(T'_j)^{\beta}x\|^p \\ &= \sum_{\substack{\beta\in\mathbb{N}^{d-1}\\ |\beta|=1,\dots,m-1}} n^{(|\beta|)}\sum_{j=1}^{d}\frac{1}{\beta!}\|T_j^{m}(T'_j)^{\beta}x\|^p + \sum_{j=1}^{d}\|T_j^n x\|^p, \quad \forall x\in X. \end{aligned} \tag{3.2}$$

That is, for all $x \in X$, for $n \geq m-1$, the sequences $n \mapsto Q^n(x)$ are almost polynomial (of degree $\leq m-1$), with the term $\sum_{j=1}^{d}\|T_j^n x\|^p$ instead of a (constant) trailing coefficient.

However, by Corollary 2.3.(i), we know that for any $x \in X$, the sequence $n \mapsto Q^n(x)$ are indeed polynomial. Since, by Proposition 2.4, for each $x \in X$, the sequence $n \mapsto \sum_{j=1}^{d}\|T_j^n x\|^p$ is bounded, we can successive compare and remove coefficients of the formula for $Q_n(x)$ as given in 2.3.(i) and (3.2), until we eventually obtain that

$$\sum_{j=1}^{d}\|T_j^n x\|^p = \|x\|^p, \quad \forall x \in X,\ \forall\ n \geq 2m-1\ . \tag{3.3}$$

Since $T_i^m T_j^m = 0$ for all $i \neq j$, by Lemma 2.6, replacing x by $T_j^{\nu}x$ with $\nu \geq m$ in this last equation, gives $\|T_j^{\nu}x\| = \|T_j^{n+\nu}x\|$ for all $n \geq 2m-1$, for all $x \in X$.

Hence, the sequences $n \mapsto \|T_j^n x\|$ become constant for $n \geq m$, for all $j \in \{1,...,d\}$, for all $x \in X$. This is 3.1.(i).

But then, (3.3) becomes

$$\sum_{j=1}^{d}\|T_j^m x\|^p = \|x\|^p, \quad \forall x \in X\ .$$

This is 3.1.(ii).

Now take any $(n_1, ..., n_d) \in \mathbb{N}^d$ with $n_j \geq m$ for all j and replace x in the equation above by $\sum_{j=1}^d T_j^{n_j}$. Then, again, since $T_i^m T_j^m = 0$ for $i \neq j$, and since $n \mapsto \|T_j^n x\|$ become constant for $n \geq m$,

$$\sum_{j=1}^d \|T_j^{m+n_j} x\|^p = \sum_{j=1}^d \|T_j^m x\|^p = \Big\| \sum_{j=1}^d T_j^{n_j} x \Big\|^p, \quad \forall x \in X.$$

Together with 3.1.(i), this implies 3.1.(iii). □

It is clear that we have a stronger result if one of the operators $T_{j_0} \in \{T_1, ..., T_d\}$ is surjective. Theorem 3.1.(i) then forces this operator to be an isometric isomorphism and by 3.1.(ii) the remaining operators are nilpotent.

If one of the operators $T_{j_0} \in \{T_1, ..., T_d\}$ is injective, by Lemma 2.6 and 3.1.(ii) we obtain at least that $T_{j_0}^m$ is an isometry and the remaining operators are nilpotent. However, while, by definition of an (m,p)-isometry, we must have $\bigcap_{j=1}^d N(T_j) = \{0\}$, it is not clear that the kernel of a single operator has to be trivial.

4 Some further remarks and the case $d = 2$

We finish this note with a stronger result for the case of a commuting pair $(T_1, T_2) \in B(X)^d$. We first state the following two easy corollaries of Theorem 3.1 which hold for general d.

Corollary 4.1. *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an (m,p)-isometry as well as a (μ,∞)-isometry. Then $T_j^m = 0$ or $\|T_j^m\| = 1$ for any $j \in \{1, ..., d\}$.*

Proof. By Theorem 3.1.(ii) we have $\|T_j^m\| \leq 1$ for any j. On the other hand, by 3.1.(i) we have

$$\|T_j^m x\| = \|T_j^{m+1} x\| = \leq \|T_j^m\| \cdot \|T_j^m x\|, \quad \forall x \in X,$$

for any j. That is, $T_j^m = 0$ or $\|T_j^m\| \geq 1$. □

Lemma 4.2. *Let $T = (T_1, ..., T_d) \in B(X)^d$ be an (m,p)-isometry as well as a (μ,∞)-isometry. Define $|.|_\infty : X \to [0,\infty)$ and $X_{j,|\cdot|_\infty}$ as in Theorem 2.5. Then*

$$X_{j,|\cdot|_\infty} = \{x \in X \mid \exists \alpha(x) \in \mathbb{N}^d, \ s.th. \ |\alpha(x)| \leq \mu - 1 \ and \\ |x|_\infty = \|T_j^n \left(T_j'\right)^{\alpha_j'(x)} x\|, \ \forall n \in \mathbb{N}\}.$$

Proof. By Proposition 2.4 we know that for every $x \in X$, there exists an $\alpha(x) \in \mathbb{N}^d$ with $\max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\| = \|T^{\alpha(x)} x\|$ and $|\alpha(x)| \leq \mu - 1$.

Then $x \in X_{j,|\cdot|_\infty}$ if, and only if, for all $n \in \mathbb{N}$, there exists an $\alpha(x,n) \in \mathbb{N}^d$ with $|\alpha(x,n)| \leq \mu - 1$ s.th. $|x|_\infty = \|T_j^n T^{\alpha(x,n)} x\|$. Hence, the inclusion "$\supset$" is clear.

To show "$\subset$" let $0 \neq x \in X_{j,|\cdot|_\infty}$. Then $T_j^m \neq 0$ and, hence, $\|T_j^m\| = 1$.

Since $|\alpha(x,n)| \leq \mu - 1$ for all $n \in \mathbb{N}$, there are only finitely many choices for each $\alpha(x,n)$. Thus, there exists an $\alpha(x) \in \mathbb{N}^d$ and an infinite set $M(x) \subset \mathbb{N}$ s.th.

$$|x|_\infty = \|T_j^n T^{\alpha(x)} x\|, \ \forall n \in M(x).$$

By Theorem 3.1.(i), $M(x)$ contains all $n \geq m$ and further,

$$\|T_j^n T^{\alpha(x)} x\| = \|T_j^n \left(T_j'\right)^{\alpha_j'(x)} x\|, \quad \text{for all } n \geq m.$$

Since $\|T_j^m\| = 1$, the statement holds for all $n \in \mathbb{N}$. □

Proposition 4.3. *Let $T = (T_1, T_d) \in B(X)^d$ be both an (m,p)-isometric and a (μ,∞)-isometric pair. Then T_1^m is an isometry and $T_2^m = 0$ or vice versa.*

Proof. By Theorem 2.5, we have $X = X_{1,|\cdot|_\infty} \cup X_{2,|\cdot|_\infty}$.

Let $x_1 \in X_{1,|\cdot|_\infty}$. Then, by the previous lemma, there exists an $\alpha_2(x_1) \in \mathbb{N}$ with $\alpha_2(x_1) \leq \mu - 1$ s.th. $|x_1|_\infty = \|T_1^n T_2^{\alpha_2(x_1)} x_1\|$ for all $n \in \mathbb{N}$.

Furthermore, we have $\|x\|^p = \|T_1^m x\|^p + \|T_2^m x\|^p$, for all $x \in X$, by Theorem 3.1.(ii). Replacing x by $T_2^{\alpha_2(x_1)} x_1$gives

$$\begin{aligned} &\|T_2^{\alpha_2(x_1)} x_1\| = \|T_1^m T_2^{\alpha_2(x_1)} x_1\| + \|T_2^{m+\alpha_2(x_1)} x_1\| \\ \Leftrightarrow\ &\|T_2^{\alpha_2(x_1)} x_1\| = |x_1|_\infty + \|T_2^m x_1\|. \end{aligned}$$

This implies $\|T_2^{\alpha_2(x_1)} x_1\| = |x_1|_\infty$ and, moreover, $\|T_2^m x_1\| = 0$.

An analogous argument shows that $X_{2,|\cdot|_\infty} \subset N(T_1^m)$. Hence,

$$X = N(T_1^m) \cup N(T_2^m),$$

which forces $T_1^m = 0$ or $T_2^m = 0$. The statement follows from $\|x\|^p = \|T_1^m x\|^p + \|T_2^m x\|^p$, for all $x \in X$. □

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