

STUDENTS' MISCONCEPTIONS CONCERNING INFINITY

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We investigate some misconceptions concerning the cardinality of infinite sets. Students from three different groups were asked to complete a short questionnaire which contained questions designed to give insight into students' concept images of infinity, cardinality and the comparison of infinite sets. The study highlighted some of the conceptions of students associated with finite, countable and uncountable infinite sets. These different conceptions and misconceptions might arise from: the meaning of words in everyday language; the incorrect use of properties which are true for some situations but not for all; the misuse of the bijection criterion; the different presentations of sets; or the idea of infinity as a number.

INTRODUCTION

Many students find the transition from computational mathematics to abstract mathematics very difficult. Therefore for mathematics lecturers at university, it is worth knowing how students learn about advanced mathematical concepts such as infinity. Teacher content knowledge has been identified as being extremely significant for successful teaching and learning; Shulman (1986) referred to this as *pedagogical content knowledge*. According to him, teachers need to understand the content they teach deeply, they need to know how the key elements of a concept might be misunderstood by students, and need to have useful ways of representing these key ideas in order to help students to overcome their misconceptions. In our view, pedagogical content knowledge is also vital for lecturers at third level, and when we speak of teachers in the remainder of this paper we include teachers at all levels. In this article we report on a preliminary case study which aims to explore students' concept images of infinity, especially their concept images of finite sets, countable infinite sets, and uncountable infinite sets.

It is known that students meet a wide range of information while they are learning mathematics, and the development of mathematical concepts will naturally depend on their previous beliefs and experience. Therefore, much work has been carried out by the mathematics education community in an effort to describe how students understand mathematical concepts. One of the most important contributions to this area is the work of Tall and Vinner (1981), which describes the distinction between the terms concept definition and concept image associated with any mathematical concept. The term *concept definition* is used to refer to a mathematical definition of the given concept; Tall and Vinner state this as "a form of words used to specify that concept" (p. 152), and the term *concept image* is used to mean all the mental pictures, the visual representations, the impressions and the experiences of the individual that are associated with the concept. Przeniolo (2004) described concept image as "the cognitive structure containing all kinds of associations and conceptions related to the concept" (p. 104). Using the ideas of Tall and Vinner (1981), many studies have been carried out to explore students understanding of a mathematical concept. Alcock and Simpson

(2009) summarised the results of many studies of this kind and also reported on students' ignorance of the status of mathematical definitions in advanced mathematics.

In most real life contexts, people acquire an understanding of a concept without the need for a proper dictionary definition, and for that reason, concept image plays an essentially important role in real life. While, on the contrary in mathematical contexts, definitions have a crucial role in the acquisition of concepts. Most mathematics teachers would expect students to base their answers on definitions when working on a problem; however students often consult only their concept image. Edwards and Ward (2004) showed that, students' inability to understand the distinction between everyday life definitions and mathematical definitions has an influence on their understanding of mathematical concepts. Przeniolo (2004) studied students' conceptions of the limits of functions, and found that students regarded their intuitions as a definition of the concept.

There are many reasons for the difficulties students face when trying to understand a mathematical concept. One of these difficulties arises when a concept has terms that are used in real life language, or when the terms have a meaning that is the opposite of the mathematical meaning. As students hold these real life conceptions for a long time, they are slow to disappear after students have been introduced to mathematical concepts that use these same everyday terms. Cornu (1991) named these conceptions of ideas that are formed from the colloquial meanings of words as *spontaneous conceptions*. He observes that many mathematical terms, such as 'limit' for example, have "a significance for students before any lessons begin, and that students continue to rely on these meanings even after they have been given a formal definition" (p. 154).

Several studies have documented students' conceptions of infinity. Since infinite iterative processes are essential to many undergraduate concepts, Dubinsky, Weller, Stinger and Vidakovic (2008) investigated students' conceptions of these processes. They applied APOS Theory (Dubinsky and McDonald 2001) to give explanations of how students might think about the infinity concept. APOS Theory aims to describe how understanding of a mathematical concept might take place. That is, conceptions are described as passing through mental stages of **A**ctions, **P**rocesses, and **O**bjects, and then are organised in **S**chema to make sense of the problem situation the individual deals with. According to APOS Theory, a mental action occurs when an individual carries out an iterative process (i.e., step-by-step). When this action is repeated, he/she can reflect upon it and imagine repeating it over time and can describe the steps without actually doing them; in this case the mental action has been interiorised to become a process. When an individual reflects on the process and can move from seeing it as carried out over time to seeing it as being carried out at a moment in time, he/she becomes aware of the process as a totality then he/she thinks of it as an object. And when the individual organises a series of objects, he/she has reached the schema level. Dubinsky et al. (2008) showed that students' difficulties with the infinity concept lay in their conception of the state at infinity as being an incomplete process; they see infinity not as an entity in its own right but as a repeated action like counting. The students had not seen the unending process as a totality; that is they had not constructed the mental object required for

APOS theory and they considered infinity as a process and not an object, and that led them to misconceptions.

We will be concerned with the attempts of students to compare infinite sets. Other authors, (Tsamir 2003) have noted that the criteria that students use to carry out this type of task include one to one correspondence, inclusion, and the notion that all infinite sets have the same cardinality. Furthermore, Tirosh (1999) reported that students intuitively resort to processes involving the elements of the set (e.g. keep adding a number, or keep subdividing a line) to determine if a given set is infinite. She also found that students sometimes determined a set to be finite by comparing it with another set which was considered to be infinite; for example they might argue that the given set is proper subset of an infinite set, so it has fewer elements than the infinite set, and hence it is finite. Tirosh and Tsamir (1996) have indicated that different representations of the same infinite sets often give rise to different answers when students are asked to compare them. They found that, representing the sets $\{1,2,3,\dots\}$ and $\{1,4,9,\dots\}$ horizontally (i.e. placing them side by side), encouraged part-whole consideration of the difference of the number of elements, that is the notion that the whole set has more elements than its subset. While arranging the sets vertically (i.e. one set over the other), triggered the use of the 1-1 correspondence criterion, and writing the sets numerically-explicitly (e.g. 1^2 appears below 1, 2^2 below 2, 3^2 below 3, etc.) encouraged justifications of 1-1 correspondence more than the vertical representation did and more often than the geometric representation which is drawing pictures of the elements of sets.

In line with the aforementioned works, we present our study to examine conceptions related to finite and infinite sets. The study showed that, students have different concept images associated with finite, countable and uncountable infinite sets and they used these in their reasoning to answer the questions we asked. We will investigate some misconceptions displayed by the students in this study.

METHODOLOGY

In this study a questionnaire was designed to investigate students' understanding concerning the infinity concept and the comparison of infinite sets. It was administered to three separate groups of students at NUI Maynooth. These three groups took three different analysis modules which were taught by the same lecturer, they all learned about infinity, cardinality and countability. The questionnaire was anonymous, participation in the study was voluntary and 35 students took part. Twelve of these students were first year students studying for a degree in Mathematics and Theoretical Physics, 13 students were out-of-field mathematics teachers who were enrolled on a postgraduate course and ten students came from a second year science class which was aimed at students who wished to study pure mathematics in their third year. For all three groups of students, the analysis module was their first exposure to rigorous mathematics. The questionnaire was given to the students during their classes and completed by them in 20 minutes. It consisted of 7 questions which explored topics concerning finite, countable and uncountable sets. The first two questions were open-ended questions to elicit students' intuitive opinions about infinity, questions 4 and 7 involved geometric sets, question 5 involved mathematical sets of numbers, and questions 3

and 6 consisted of sets from real life. Most of the questions on the questionnaire were employed in previous studies. In some cases we modified the questions, and in other cases we changed the situation that was presented to a situation that was applicable in our work. A copy of the questionnaire can be found in the appendix.

The method we used in our analysis was qualitative because we aimed to gather a deep understanding of students thinking, so the analysis we will present is based on students' responses to the questions we asked and in particular the reasons they gave for their responses. We started by analysing each student in each group separately and in order to obtain more information on the concept images students hold for finite and infinite sets the analysis was first carried out individually. We kept note of the misconceptions the students have and we met several times to discuss the results of each student and then each group. We discussed what we found many times to reveal the similarity, difference and dominance of students' conceptions in all groups. Then we classified the conceptions of each group into categories and then we gathered the categories of all groups until we reached our final results concerning the different kinds of conceptions students have regarding the infinity concept.

RESULTS

The primary analysis which we carry out here is based on the types of arguments written by the students to justify their answers to the problems on the questionnaire. Our main concern in this article was to see how students regard finite, countable and uncountable infinite sets. We tried to explore the dominant concept images which the students used most frequently. Our results showed that, there are different types of misconceptions related to finite sets and the countability of infinite sets. We break these misconceptions down into five types: misconceptions based on daily experience; misconceptions based on the misuse of properties; misconceptions concerning the use of the bijection criterion; misconceptions based on the understanding of sets; and misconceptions concerning the idea of infinity. We will explain each type as follows.

Misconceptions Based on Daily Experience

Daily experience can affect students' conceptions, especially when the terms being used in a mathematical concept have a different meaning in everyday life. Cornu (1991) called those conceptions "spontaneous conceptions". Our study showed that, many students hold those kinds of conceptions. The term 'countable' has a significant meaning in English that is 'can be physically counted'. Some students use the terms finite and countable interchangeably. The notion that the elements of a finite set can be counted and so the set is countable can be seen in the responses to Question 3:

M is finite as all the melodies that have been composed are a set number, they are countable. (1.8)

This spontaneous conception could also lead students to think that an infinite set is a set such that its elements are uncountable or can not be counted, and we can see this clearly in this argument:

H is infinite as since there is no way of knowing how long the earth will go on for, and how many melodies will be composed. We must assume that this number is uncountable, and therefore H is infinite. (1.5)

The practice of using the terms ‘countable’ and ‘uncountable’ for the elements of a set and not for the set itself might be a reason for other students to assume that all infinite sets are uncountable; rather than all uncountable sets are infinite. A student responded to Question 5:

Yes, A is equivalent to \mathbb{N} , because there is an infinite number of elements in both sets and both are uncountable. (T.11)

Students seem to associate the word ‘cardinality’ with the number of elements of a set and this seems to foster the belief that the cardinality of uncountable sets is unknown (since we cannot count the elements of the set). For example, in response to Question 7 one student said:

\overline{AB} and $CDEF$ are uncountable sets, so we don’t know their cardinality. Therefore we cannot say which is bigger. (2.1)

We also found that some students used the phrase ‘can be counted’ in relation to both finite sets and countable infinite sets. Those students who hold these conceptions seem to contradict themselves, for instance one student commented according to Question 3(a) that:

M is finite. As there is a number of tunes, we can count them. (2.1)

And according to Question 6(a) the same student commented that:

Both sets G and P are countable as we can count them: $G := \{1,2,3,\dots\} = \mathbb{N}$ and $P := \{1,2,3,\dots\} = \mathbb{N}$ (2.1)

Counting seems to be used in two different ways here.

Misconceptions Based on the Misuse of Properties

Tirosh (1991) found that many students assumed incorrectly that “all methods suitable for comparing finite sets are adequate for infinite sets as well” (p. 204). Our results showed that many students seem to have misconceptions that arise from using theorems or properties that have been shown to hold true for countable infinite sets when considering uncountable sets. We sort these misconceptions into two types. Firstly, it is true that, every subset of a countable set is countable. Some students incorrectly assume that every subset of an uncountable set is uncountable. Students’ justifications in Questions 5 and 7 were evidence of these conceptions. The main argument used to justify their answers was: A is not equivalent to \mathbb{N} because A is a subset of R so it is an uncountable infinite set. For instance, regarding Question 5 this student claimed that:

No, \mathbb{N} countable, the set A is uncountable as it’s a subset of uncountable set R (1.7).

Another student answering Question 7 argued that:

Yes, both are subsets of R and equivalent to R so therefore equal (T.5).

We can see that the reason used to justify the answers above is incorrect. \mathbb{N} is also a subset of R, but those students in this situation did not think of that, and they seem only to remember

that N is a subset of R if we ask them the question directly. They seem to have a conception that R contains only intervals or line segments.

Secondly, it is true that all infinite countable sets are numerically equivalent to each other, but in fact we found that many students assume also that all uncountable infinite sets are equivalent. We notice these conceptions most obviously in the arguments with respect to Question 7. The most used argument here was that all the sets are uncountable; therefore they have the same cardinality. For example:

Yes, The line \overline{AB} contains an uncountably infinite number of points and the square's cardinality is uncountably infinite, and as result, numerically equivalent (1.4).

Another student commented that:

Yes, Unsure, on how to explain. Need to look (to) my notes!

Idea \rightarrow both AB and $CDEF \rightarrow$ Uncountable (T.4).

Misconceptions Based on the Misuse of the Bijection Criterion

The students have learned in the course that the bijection criterion is the main criterion that should be used to compare infinite sets. But our findings showed that many students did not use this criterion to determine their responses in the four problems that dealt with comparison of infinite sets. Of the students who used this criterion, none of them used it in all comparisons of infinite sets. The most use was found in their answers to Question 6, maybe because the sets in that question are countable and the elements can be seen clearly and therefore the bijection between these sets is more obvious. We also found that students who used the bijection criterion used it correctly in some problems and incorrectly in other problems. To illustrate an incorrect use of bijection, a student when answering Question 5 thought that since both sets are infinite then there must be a bijection between them; hence he or she claimed that:

Yes, they are both infinite sets. A bijection will map $N \rightarrow A$. (1.1)

Even when students invoked the bijection criterion they rarely wrote down a specific map. Some students made diagrams for their answers to Question 6 to illustrate the bijection, and others used other informal terms rather than bijection (i.e. you can match/pair up the elements of the comparison sets), like:

Yes, for each glass, there is a plate. They can be paired up. (1.8)

Similarly, when trying to use the bijection criterion to show sets were not equivalent, students did not put forward an argument as to why a bijection could not exist but just stated the fact. For example in answer to Question 7 one student claimed:

No, each point in the square cannot map directly 1-1 onto the line (T.8).

Misconceptions Based on the Understanding of Sets

We found that some students were unable to write the sets F and K which are given in Question 6 (b) and Question 6 (c) in a mathematical way and that led them to use incorrect justifications for their claims. The elements of sets F and K are difficult to write down and

require the use of some notation. However, many students tried to write them as subsets of \mathbb{N} . For instance (note E below denotes the even numbers):

No, As $F = \{2,4,6,8,\dots\} = \text{Even no.'s}$, $G = \mathbb{N}$, \mathbb{N} is (supposedly!) bigger than E (1.2).

A different student expressed F incorrectly in Question 6(b), however he or she concluded that a bijection could exist between G and F and argued that:

Yes, $G = \{1,2,3,\dots\}$, $F = \{2,4,6,\dots\}$. $\# G = \# \mathbb{N}$. There is a bijection from G to F (1.10).

These students seem to grasp the essential ideas concerning countable sets but their inability to translate the description of the sets into mathematical notation hinders their efforts.

In line with this, many students interpreted the set A in Question 5 incorrectly; they thought of $(1.25, 3.79)$ as $\{1.25, 3.79\}$ and therefore for them A seems to contain only 2 numbers and thus is not equivalent to \mathbb{N} . Some of them argued that:

No, the set A only has 2 numbers; therefore all the natural numbers cannot biject onto it. (T.8)

Perceptions of the Concept of Infinity

For a long time the concept of infinity has been a cause of debate for many philosophers and mathematicians, so it is no wonder that students also have difficulties with understanding the notion of infinity. Many of the difficulties with infinity seem to arise when students think of infinity as a number, albeit one that is not reachable. In the questionnaire, students were asked to explain the idea of infinity and many of them thought of it as an unreachable number, for example:

Representation of a number that [can] never be reached as there will always be more numbers bigger (1.7).

Another student thought of infinity as the largest number, but one that does not exist:

Infinity is the highest possible number, but by the same token, it does not exist. It means there is no such largest number (1.2).

It is endlessness, not number itself but the largest natural number can tend towards (1.9).

Another student mentioned that:

I would say it as the property of numbers that they do not have an end or largest value, and we call infinity the theoretical largest number (2.2).

Another one commented that:

I would say it is a concept. Infinity may not really exist in concrete terms. It is the idea that if you counted forever, the "last number" would be infinity. Or more realistically, you would say the last number is infinity as you would never be able to actually reach it (2.9).

The view that infinity is a number might be responsible for students thinking that all infinite sets are equivalent as all have an infinite number of elements. To show that, a student wrote regarding Question 4 (b):

No, $\text{Infinity} + \text{Infinity} = \text{Infinity}$ and $\text{Infinity} = \text{Infinity}$ (2.9).

Another student commented that:

No, Since set AB has an infinite amount of points, CD can not contain more elements than an infinite amount (1.9).

A different student supposed that there is only one kind of infinity and argued that:

Infinity cannot be greater than infinity, they are of equal cardinality. You could also probably get a bijection from one set to the other (1.5).

CONCLUSIONS

In this study, we aimed to examine students' conceptions about the infinity concept, and their methods of comparing finite sets, countable sets and uncountable infinite sets. From our findings we see that students hold many different types of conceptions related to infinite sets. We found that everyday language is one reason for students' misunderstandings. Many of our students supposed that a countable set is a set for which its elements could be (physically) counted (from the real life use of the word), and by this meaning a countable set is a finite set. Similarly for these students an infinite set means an uncountable set. Moreover in accordance with Tirosh's (1991) study in which many students used methods that are applicable for finite sets for infinite sets also, in our study we found that many students supposed properties that are true for countable sets to be true for uncountable sets also. That is, they think that any subset of an uncountable set is also uncountable, and all uncountable sets are equivalent. Furthermore the study indicated that although some students understand that the bijection criterion is the main method to determine the comparison of infinite sets, they still thought that all infinite sets are equivalent and were unable to apply the bijection criterion correctly. In addition, we found that some students were not able to express real life sets mathematically in a correct way; for example they tried to express F the set of forks as $F = \{2,4,6,\dots\}$. We can see that they tried to use 2 here to indicate 2 elements (forks), but writing F in this way caused some of them to think that F has fewer elements than N. The problems our students had with representations of sets echo the findings of Tirosh and Tsamir (1996). What is more, students understanding of infinity as a number that is unreachable might be a reason to think that all infinities are the same and all sets with infinite cardinalities are equivalent.

Most of the difficulties that the students in this study encountered with the concept of infinity have been reported in previous studies (see Tirosh (1991) for an overview). However, we have not been able to find other studies that mention student's difficulties with the spontaneous conception related to the word 'countable'.

In conclusion, we feel that a mathematics teacher or lecturer must be aware of the misconceptions related to any specific concept they teach, and it is beneficial for them to know most, if not all, difficulties students might encounter with it. We hope our results contribute to give some view of these difficulties with the concept of infinite sets.

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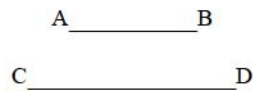
APPENDIX

The Questionnaire problems:

1. How would you explain the idea of infinity to a friend of yours?
2. Why is \mathbb{N} infinite?
3. Let M be the set of all melodies (tunes) that have been composed until now. Let H be the set of all melodies that could be composed.
 - (a) Is M finite or infinite? Explain!

(b) Is H finite or infinite? Explain!

4. Look at the line segments AB and CD below:



Answer the following by putting a circle around the correct reply:

(a) Is the set of points on AB finite or infinite?

Finite Infinite Don't know

Explain your answer!

(b) Is the cardinality of the set CD greater than the cardinality of the set AB?

Yes No Don't know

Explain your answer!

5. Compare the set $A = (1.25, 3.79)$ with N the set of natural numbers. Are they numerically equivalent?

Yes No Don't know

Explain your answer!

6. An infinite dinner table is set in a restaurant. Each person is served a glass, a plate, three knives and two forks. Let F be the set of forks on the table, K be the set of knives on the table, G be the set of glasses on the table, and P the set of plates on the table.

(a) Is the cardinality of G equal to the cardinality of P ?

Yes No Don't know

Explain your answer!

(b) Is the cardinality of F equal to the cardinality of G ?

Yes No Don't know

Explain your answer!

(c) Is the cardinality of F equal to the cardinality of K ?

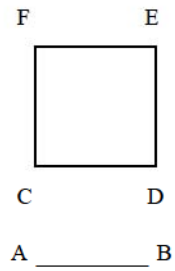
Yes

No

Don't know

Explain your answer!

7. Look at the diagram below.



Is the line segment AB numerically equivalent to the square CDEF?

Yes

No

Don't know

How did you come to this conclusion?