LOCALNESS OF $A(\Psi)$ ALGEBRAS

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ABSTRACT

Let d and r be positive integers. Given $\Psi = (\psi_1, \ldots, \psi_r) \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$, we consider the unital algebra $\mathbf{R}[\Psi] = \mathbf{R}[\psi_1, \ldots, \psi_r]$ generated by $\{\psi_1, \ldots, \psi_r\}$, and its closure $A(\Psi)$ in C^∞ topology.

We identify the space of closed maximal ideals of $A(\Psi)$, we establish that it is a regular algebra, and we show that the approximation problem, to provide an explicit description of Ψ, is local to the level sets of Ψ.

1. Introduction.

Let d and r be positive integers, throughout. Given $\Psi = (\psi_1, \ldots, \psi_r) \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$, we consider the unital algebra $\mathbf{R}[\Psi] = \mathbf{R}[\psi_1, \ldots, \psi_r]$ generated by $\{\psi_1, \ldots, \psi_r\}$, and its closure $A(\Psi)$ in C^∞ topology.

We recall that $A(\Psi)$ is also the closure of the algebra

$$C^\infty(\Psi) = \{g \circ \Psi : g \in C^\infty(\mathbf{R}^r, \mathbf{R})\}$$

(cf. [1,2] for background).

This paper is about the properties of $A(\Psi)$ *qua* Fréchet algebra, and the approximation problem: describe $A(\Psi)$ explicitly.

We identify the space of closed maximal ideals of $A(\Psi)$, we establish that it is a regular algebra, and we show that the approximation problem is local to the level sets of Ψ.

2. Associated Topologies.

We shall only use the Euclidean topology on $\mathbf{R}^r$, but we need to consider some other, *a priori* distinct topologies on $\mathbf{R}^d$.

Definition. We define the Ψ–*hull–kernel topology* on $\mathbf{R}^d$ as that corresponding to the Kuratowski closure operation

$$E \mapsto \mathrm{HK}(E) = \{b \in \mathbf{R}^d : f(b) = 0 \text{ whenever } f \in A(\Psi) \text{ and } f|E = 0\}.$$

Remark. This is in general finer than the pull–back topology

$$\{\Psi^{-1}(U) : U \text{ open in } \mathbf{R}^r\}.$$

It may happen that $\mathrm{clos}\Psi(E) \cap \mathrm{clos}\Psi(F) \neq 0$ for disjoint Ψ–hull–kernel closed sets E, F.

We abbreviate Ψ–hull–kernel topology to HK–topology when convenient. We denote the HK–closure of a set E by $\mathrm{HK}(E)$. This is consistent with the following (more–or–less standard) notation:

$$H(F) = \{a \in \mathbf{R}^d : f(a) = 0, \forall f \in F\}, \forall F \subset A(\Psi);$$

$$K(E) = \{f \in A(\Psi) : f(a) = 0, \forall a \in E\}, \forall E \subset \mathbf{R}^d.$$

Note that

$$HKH(F) = H(F), \forall F \subset A(\Psi),$$

$$KHK(E) = K(E), \forall E \subset \mathbf{R}^d.$$

In particular, $H(F)$ is HK-closed, for each $F \subset A(\Psi)$.

We note that each HK–closed set is a union of level sets of Ψ, that HK($\{a\}$) = $\Psi^{-1}(\Psi(a))$ (the level set through the point a), and that the minimal nonempty HK–closed sets are these level sets of Ψ. We abbreviate HK($\{a\}$) to HK(a).

Definition. We define the Ψ–*weak–star topology* on $\mathbf{R}^d$ as the pull–back topology corresponding to the weak–star topology on the dual $A(\Psi)^*$ and the natural injection of $\mathbf{R}^d$ into $A(\Psi)^*$.

In other words, the set N is a weak–star neighbourhood of the point $a \in \mathbf{R}^d$ if and only if there exist a finite number of functions $f_1, \ldots, f_n$ belonging to $A(\Psi)$, such that

$$\{x \in \mathbf{R}^d : |f_j(x) - f_j(a)| < 1, \forall j\} \subset N.$$

Since we are dealing here with real–valued functions, it is evident that it makes no difference if we insist that n always equal 1. In fact, it is easy to see that the set N is a weak–star neighbourhood of the point $a \in \mathbf{R}^d$ if and only if there exists a function $f \in A(\Psi)$, such that $f(a) = 0$ and

$$\{x \in \mathbf{R}^d : f(x) < 1\} \subset N.$$

We abbreviate Ψ–weak–star topology to WS–topology, when convenient.

Unqualified topological terms (open, closed, ...) refer to the Euclidean topology. It is clear that the WS–topology is at least as fine as the HK–topology, and the Euclidean topology is at least as fine as the WS–topology. We shall discover more below.

We denote the closed ball with centre $x \in \mathbf{R}^d$ and radius $r \geq 0$ by $\mathbf{B}(x, r)$, and the corresponding open ball by $\mathbf{U}(x, r)$.

Proposition 1. *Let $\Psi \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$. Suppose E and F are disjoint closed subsets of $\mathbf{R}^d$, $\Psi^{-1}\Psi(E) = E$, and $\Psi^{-1}\Psi(F) = F$. Then there exists $f \in A(\Psi)$ such that $f = 1$ on E and $f = 0$ on F.*

PROOF. Let E and F be as in the hypothesis. Define

$$B_n = \mathbf{B}(0, n), \qquad B'_n = \Psi(B_n),$$

$$E_n = E \cap B_n, \qquad E'_n = \Psi(E_n),$$

$$F_n = F \cap B_n, \qquad F'_n = \Psi(F_n),$$

$$H_n = B_{n-1} \cup E_n \cup F_n, \qquad H'_n = \Psi(H_n),$$

We observe that E'_n,F'_n, and H'_n are compact subsets of $\mathbf{R}^r$, and

$$E'_n \cap F'_n = \emptyset,$$

$$E_{n+1} \cap B_n = E_n,$$

$$E'_{n+1} \cap B'_n = E'_n, \qquad \text{(because } E = \Psi^{-1}\Psi(E))$$

$$F'_{n+1} \cap B'_n = F'_n.$$

Choose $g_1 \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that $g_1 = 1$ near E'_1 and $g_1 = 0$ near F'_1.

Consider the function $g_2 : H'_2 \to \mathbf{R}$ given by

$$g_2(x) = \begin{cases} g_1(x), x \in B'_1, \\ 1, x \in E'_2, \\ 0, x \in F'_2, \end{cases}$$

This is well–defined, since $g_1 = 1$ on $B_1' \cap E_2'$ and $g_1 = 0$ on $B_1' \cap F_2'$. Moreover, each point of H_2' has a neighbourhood to which g_2 has a C^∞ extension; indeed one of g_1, 1 or 0 will do as the extension. Since the existence of a global C^∞ extension is a local property, it follows that g_2 has an extension in $C^\infty(\mathbf{R}^r, \mathbf{R})$, and we denote such an extension by the same symbol, g_2. We may choose g_2 so that it is 1 *near* E_2' and 0 *near* F_2'.

Continuing in this way, we find $g_{n+1} \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that

$$g_{n+1}|B_n' = g_n|B_n',$$

$g_{n+1} = 1$ near E_{n+1}' and $g_{n+1} = 0$ near F_{n+1}'.

Define $f : \mathbf{R}^d \to \mathbf{R}$ by setting

$$f|B_n = g_n \circ \Psi|B_n, \forall n.$$

Evidently, f is well–defined, $f \in C^\infty(\mathbf{R}^d, \mathbf{R})$, and $g_n \circ \Psi \to f$ in C^∞ topology as $n \uparrow +\infty$. Thus $f \in A(\Psi)$. Finally, it is clear that $f = 1$ on E and $f = 0$ on F, so we are done. ∎

Corollary 2. *Suppose $E \subset \mathbf{R}^d$. Then E is HK–closed if and only if E is closed and $\Psi^{-1}\Psi(E) = E$.*

PROOF. The 'only if' part is obvious. For the converse, suppose that E is closed and $\Psi^{-1}\Psi(E) = E$. Let $a \notin E$. Then $F = \Psi^{-1}\Psi(a)$ is closed and disjoint from E, and $\Psi^{-1}\Psi(F) = F$. By the Proposition, there exists $f \in A(\Psi)$ such that $f|E = 0$ and $f|F = 1$. Thus $a \notin \mathrm{HK}(E)$. This shows that $\mathrm{HK}(E) \subset E$. Evidently $E \subset \mathrm{HK}(E)$, so $\mathrm{HK}(E) = E$, and we are done. ∎

Corollary 3. *Let $E \subset \mathbf{R}^d$. Then HK(E) is the least set $F \subset \mathbf{R}^d$ such that $E \subset F$, F is closed, and $\Psi^{-1}\Psi(F) \subset F$.* ■

This fact implies that the HK-closure of a set E may be obtained by forming $E_0 = E$, and proceeding by transfinite induction:

$$E_{\alpha+1} = \text{clos}\Psi^{-1}\Psi(E_\alpha), \forall \text{ ordinals } \alpha,$$

$$E_\alpha = \bigcup_{\beta<\alpha} E_\beta, \forall \text{ limit ordinals } \alpha,$$

until the first ordinal having cardinal greater than the continuum, at the worst.

Corollary 4. *Let $\Psi \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$. Then the following three conditions are equivalent:*

(1) Ψ is injective.

(2) The HK–topology is Hausdorff.

(3) The HK–topology is the same as the Euclidean topology.

PROOF. Obviously (3) $\Rightarrow$ (2) $\Rightarrow$ (1). The only delicate point is (1) $\Rightarrow$ (3), and this is immediate from Corollary 2. ■

Finally, we note that combining Proposition 1 and Corollary 2, we have:

Corollary 5. *Let E and F be disjoint HK–closed sets. Then there exists $f \in A(\Psi)$ such that $f = 1$ on E and $f = 0$ on F.* ■

Now we consider the C^∞ action and its consequences.

Proposition 6. *Let $h \in C^\infty(\mathbf{R}^r, \mathbf{R}^p)$. Then the (usually nonlinear) map*

$$h\circ : \begin{cases} C^\infty(\mathbf{R}^d, \mathbf{R}^r) \to C^\infty(\mathbf{R}^d, \mathbf{R}^p), \\ \qquad\qquad f \mapsto h \circ f, \end{cases}$$

is continuous.

PROOF. This is immediate from Faa di Bruno's formula [4, p.222]. ∎

Corollary 7. *$C^\infty(\mathbf{R}, \mathbf{R})$ acts by composition on $A(\Psi)$, i.e. $f \mapsto h \circ f$ maps $A(\Psi)$ into itself, whenever $h \in C^\infty(\mathbf{R}, \mathbf{R})$.*

PROOF. Let $f \in A(\Psi)$. Choose $g_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that $g_n \circ \Psi \to f$ in $C^\infty(\mathbf{R}^d, \mathbf{R})$ topology. By the Proposition,

$$(h \circ g_n) \circ \Psi = h \circ (g_n \circ \Psi) \to h \circ f$$

in C^∞ topology. Thus $h \circ f \in A(\Psi)$. ∎

Corollary 8. *Let $h \in C^\infty(\mathbf{R}, \mathbf{R})$, with $h(0) = 0$. Then $h\circ$ maps each ideal $I \subset A(\Psi)$ into itself.*

PROOF. We may factorize $h(x)$ as $xk(x)$, where $k \in C^\infty(\mathbf{R}, \mathbf{R})$[T].

Let $f \in I$. Then, using Corollary 2, we get

$$h \circ f = f \cdot (k \circ f) \in f \cdot A(\Psi) \subset I,$$

as required. ∎

Corollary 9. *Let $g \in A(\Psi)$. Then the set*

$$U = \{a \in \mathbf{R}^d : g(a) > 0\}$$

is HK–open.

PROOF. Choose $h \in C^\infty(\mathbf{R}, \mathbf{R})$ such that $h(x) > 0$ whenever $x > 0$ and $h(x) = 0$ whenever $x \leq 0$.

By Corollary 7, $h \circ g \in A(\Psi)$, and evidently

$$\mathbf{R}^d \sim U = H(\{h \circ g\})$$

is HK–closed. This suffices. ∎

Corollary 10. *The HK–topology is the same as the WS–topology.*

PROOF. It follows readily from Corollary 9 that the set

$$\{x \in \mathbf{R}^d : f(x) < 1\}$$

is HK–open, whenever $f \in A(\Psi)$. Since the sets of this form, corresponding to $f \in A(\Psi)$ with $f(a) = 0$, form a neighbourhood base for the point $a \in \mathbf{R}^d$, we conclude that each WS–open set is HK–open, and this suffices. ∎

We summarize our characterizations of the HK–topology, adding a useful converse to Corollary 9:

Theorem 11. *Let $U \subset \mathbf{R}^d$. Then the following are equivalent:*

(1) U is HK–open.

(2) U is WS–open.

(3) U is open and $\Psi^{-1}\Psi(U) = U$.

(4) There exists $g \in A(\Psi)$ such that

$$U = \{x \in \mathbf{R}^d : g(x) > 0\}.$$

PROOF. In view of Corollaries 2, 9 and 10, it only remains to prove that (1) implies (4).

Let $F = \mathbf{R}^d \sim U$.

Given $n \in \mathbf{N}$, consider $B_n = \mathbf{B}(0, n)$ and

$$E_n = \left\{ x \in B_n : \operatorname{dist}(x, F) \geq \frac{1}{n} \right\}.$$

For each $a \in E_n$, there exists $f_a \in A(\Psi)$ such that $f(a) = 1$ and $f = 0$ on F. Thus the set

$$N_a = \{x \in \mathbf{R}^d : f_a(x) > 0\}$$

is a HK–neighbourhood of a. Since E_n is compact, we may choose $a_1,a_2,\ldots,a_m \in E_n$ such that $N_{a_1},\ldots,N_{a_m}$ cover E_n. Let

$$g_n = f_{a_1}^2 + \cdots + f_{a_m}^2.$$

Then $g_n \in A(\Psi)$, $g_n = 0$ on F, $g_n \geq 0$ on $\mathbf{R}^d$, and $g_n > 0$ on E_n. Let

$$M_n = 1 + \max_{|i| \leq n} \sup_{B_n} |\partial^i g_n|.$$

Define

$$g = \sum_{n=1}^{\infty} \frac{g_n}{2^n M_n}.$$

For any given $m \in \mathbf{N}$ and $k \in \mathbf{Z}_+$, we have

$$\sup_{B_m} \left| \partial^i \left(\frac{g_n}{2^n M_n} \right) \right| \leq 2^{-n}, \forall |i| \leq k \quad \forall n \geq m.$$

Thus the series for g converges in C^∞ topology, and $g \in A(\Psi)$.

Evidently, $g > 0$ on U and $g = 0$ on F, so we are done. ∎

3. Maximal Ideals.

Proposition 12. *Suppose $f \in A(\Psi)$ and $f(a) \neq 0$, $\forall a \in \mathbf{R}^d$. Then $1/f \in A(\Psi)$.*

PROOF. Fix K compact in $\mathbf{R}^d$ and $k \in \mathbf{Z}_+$. It suffices to show that there exist $g_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that

$$\partial^i (g_n \circ \Psi) \to \partial^i \left(\frac{1}{f} \right), \forall |i| \leq k,$$

uniformly on K.

We may assume that K is a ball, without loss in generality.

Choose $h_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that

$$\partial^i (h_n \circ \Psi) \to \partial^i (f), \forall |i| \leq k,$$

uniformly on K. Since $f \neq 0$ on K, we may assume, without loss in generality, that $f > 0$ on K (since K is connected). Let $\kappa = \inf_K f$.

Discarding some initial terms of the sequence (if need be), we may assume that $h_n > \kappa/2$ on $\Psi(K)$, for each n.

Choose $k_n \in C^\infty(\mathbf{R}^r, \mathbf{R})$ such that $k_n = \log h_n$ near K, and let $r_n = \exp(-k_n)$. Then $g_n = 1/h_n$ on $\Psi(K)$, and

$$\partial^i (g_n \circ \Psi) = \partial^i \left(\frac{1}{h_n \circ \Psi} \right) \to \partial^i \left(\frac{1}{f} \right), \forall |i| \leq k,$$

uniformly on K, as required. ■

Theorem 13. *Let $M \subset A(\Psi)$. Then the following are equivalent:*

(1) M is a closed maximal ideal in $A(\Psi)$.

(2) There exists $a \in \mathbf{R}^d$ such that $M =$K(a).

PROOF. It is easy to see that (2) implies (1).

To prove that (1) implies (2), fix a closed maximal ideal M. We wish to show that M is the HK-closure of some point. Since (2) implies (1), it suffices to show that H(M) is nonempty.

Suppose that H$(M) = \emptyset$.

For each $a \in \mathbf{R}^d$, we may choose $f_a \in M$ such that $f_a(a) = 1$. Using compactness, as in the proof of Proposition 12, we may choose for each $n \in \mathbf{N}$ a function $g_n \in M$ such that $g_n \geq 0$ on $\mathbf{R}^d$ and $g_n > 0$ on B_n. Adding these up with suitable weights, we get $g \in$clos$(M) = M$ such that $g > 0$ on $\mathbf{R}^d$. By Proposition 12, $1/g \in A(\Psi)$, so $1 \in M$, so $M = A(\Psi)$, contradicting the maximality of M. This contradiction shows that H$(M) \neq \emptyset$, and we are done. ■

Remark. The argument of this proof actually shows that each ideal having empty hull is dense in $A(\Psi)$. From this observation, it is not hard to deduce that the maximal closed ideals are the same as the closed maximal ideals.

Example. Let $\Psi(x) = x$, $\forall x \in \mathbf{R}$, so that $A(\Psi) = C^\infty(\mathbf{R}, \mathbf{R})$. The subset of all functions having compact support is an proper ideal, and hence is contained in a maximal ideal M.

Since $\mathrm{H}(M) = \emptyset$, M cannot be closed.

In fact, this $A(\Psi)$ has many dense maximal ideals, corresponding to some kind of ultrafilters.

Recall that a *character* of a real Fréchet algebra is, by definition, a nonzero algebra homomorphism from the algebra to $\mathbf{R}$. There is a bijective correspondence between characters and maximal ideals with quotient isomorphic to $\mathbf{R}$. It is known that all characters on real Fréchet algebras are necessarily continuous [3].

Corollary 14. *The characters of $A(\Psi)$ are the evaluations at the points of $\mathbf{R}^d$.*

PROOF. Since characters are continuous, the kernel of a character is a closed maximal ideal, hence is $\mathrm{K}(a)$ for some $a \in \mathbf{R}^d$. It follows easily that the character is evaluation at a. ∎

Characters belong to $A(\Psi)^*$, so the space of characters inherits the weak–star topology. We may thus rephrase Corollary 5, as follows:

Theorem 15. *Let E and F be disjoint weak–star closed sets of characters on $A(\Psi)$. Then there exists a function $f \in A(\Psi)$ such that $\phi(f) = 1$ for all $\phi \in E$ and $\phi(f) = 0$ for all $\phi \in F$.* ∎

This is the *regularity* referred to in the introduction.

4. Localness.

Segal asked in 1949 whether $A(\Psi)$ has a local description analogous to the Stone–Weierstrass Theorem. Nachbin conjectured that membership of f in $A(\Psi)$ is determined by the behaviour of f on each level set of Ψ. This conjecture may be reformulated in in terms of Taylor series, and some special cases have been proved by Tougeron and the authors. For a more detailed account of the history, see [1, 2]. To date, it has not even been established that membership in $A(\Psi)$ depends only on the behaviour of f *near* each level set of Ψ. This we shall now do.

First, we establish a preliminary fact.

Lemma 16. *Let $E \subset U \subset \mathbf{R}^d$, where E is HK–closed and U is (Euclidean) open. Let K be compact. Then there exists a HK–open set V such that $E \subset V$ and $K \cap V \subset K \cap U$.*

PROOF. By Theorem 11, we may choose $h \in A(\Psi)$ such that $h = 0$ on E and $h > 0$ off E. Let $\eta = \inf_{K \sim U} h$. Then $\eta > 0$. Take $V = \{x \in \mathbf{R}^d : h(x) < \eta\}$. Then V is HK–open, by Corollary 9, $E \subset V$, and $K \cap V \subset K \cap U$, as required. ∎

Theorem 17. *Let $\Psi \in C^\infty(\mathbf{R}^d, \mathbf{R}^r)$ and $f \in C^\infty(\mathbf{R}^d, \mathbf{R})$. Then the following four conditions are equivalent:*

(1) $f \in A(\Psi)$;

(2) $\forall a \in \mathbf{R}^d$, there exists a HK-neighbourhood U of a and there exists a function $g \in A(\Psi)$ such that $g = f$ on U.

(3) $\forall a \in \mathbf{R}^d$, there exists a HK-open neighbourhood U of a and there exists a sequence of $g_n \in \mathbf{R}[\Psi]$ such that $g_n \to f$ in $C^\infty(U, \mathbf{R})$ topology.

(4) $\forall a \in \mathbf{R}^d$*, and for each compact* $K \subset \mathbf{R}^d$*, there exists a (Euclidean) open neighbourhood* U *of* $K \cap HK(a)$ *and there exists a sequence of* $g_n \in \mathbf{R}[\Psi]$ *such that* $g_n \to f$ *in* $C^\infty(U, \mathbf{R})$ *topology.*

PROOF. It is evident that (1) implies (2), (2) implies (3), and (3) implies (4).

(4) $\Rightarrow$ (1): Suppose (4). Fix $K \subset \mathbf{R}^d$, compact, and $k \in \mathbf{Z}_+$.

For each $a \in K$, choose an open neighbourhood U_a of $K \cap \mathrm{HK}(a)$ and a sequence $g_{a,n} \in \mathbf{R}[\Psi]$ such that $g_{a,n} \to f$ in $C^\infty(U_a, \mathbf{R})$ topology.

For each $a \in K$, Lemma 16 allows us to choose a HK–open set V_a such that $\mathrm{HK}(a) \subset V_a$ and $K \cap V_a \subset K \cap U_a$.

By Corollary 5 we may choose $\rho_a \in A(\Psi)$ such that $\rho_a > 0$ precisely on V_a and $\rho = 1$ on $\mathrm{HK}(a)$. Let $W_a = \{x \in \mathbf{R}^d : \rho_a(x) > 1/2\}$. Then W_a is HK–open and $\mathrm{HK}(a) \subset W_a$.

By compactness, we may choose $a_1, \ldots, a_m \in K$ such that $K \subset W_{a_1} \cup \cdots \cup W_{a_m}$. Let us abbreviate W_{a_i} to W_i, V_{a_i} to V_i, and $g_{a_i,n}$ to $g_{i,n}$. Choose $h_i \in A(\Psi)$ such that $h_i > 0$ precisely on W_i.

Let $W = W_1 \cup \cdots \cup W_m$. Since W is HK–open, we may choose $h \in A(\Psi)$ such that $h > 0$ precisely on W. Since K is compact, the number $\eta = \inf_K h$ is strictly positive. Let $F = \{h \geq \eta/2\}$. Then F and $\mathbf{R}^d \sim W$ are disjoint HK–closed sets, and $K \subset F$. Choose $h_0 \in A(\Psi)$ such that $h_0 > 0$ precisely on $\mathbf{R}^d \sim F$. Then $s = h_0 + \cdots + h_m$ belongs to $A(\Psi)$ and $s > 0$ on $\mathbf{R}^d$, so $1/s \in A(\Psi)$, by Proposition 12. Let $k_i = h_i/s$. Then $k_i \in A(\Psi)$, $k_i \geq 0$, $\sum_0^m k_i = 1$, $k_0 = 0$ near K and $\mathrm{spt} k_i \subset W_i$ whenever $i \geq 1$.

Fix i, $1 \leq i \leq m$. Let $T_i = K \cap \{\rho_i \geq 1/2\}$. Then T_i is a compact subset of $K \cap U_i$.

Since $k_i = 0$ on $K \sim T_i$, and $g_{i,n} \to f$ in $C^\infty(U_i, \mathbf{R})$, we see that

$$\partial^i (k_i \cdot g_{i,n} - k_i \cdot f) \to 0$$

uniformly on K, for each $|i| \leq k$, as $n \uparrow \infty$. Since $k_0 = 0$ near K, we conclude that the function $r_n = \sum_1^m k_i g_{i,n}$, which belongs to $A(\Psi)$, converges uniformly, along with all derivatives up to order k, uniformly on K, to $\sum_1^m k_i f$, which equals f on K. This suffices to show that (1) holds. ■

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