

# Infima of Universal Energy Functionals on Homotopy Classes

Stefan Bechtluft-Sachs

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Abstract

*We consider the infima  $\tilde{E}(f)$  on homotopy classes of energy functionals  $E$  defined on smooth maps  $f: M^n \rightarrow V^k$  between compact connected Riemannian manifolds. If  $M$  contains a submanifold  $L$  of codimension greater than the degree of  $E$  then  $\tilde{E}(f)$  is determined by the homotopy class of the restriction of  $f$  to  $M \setminus L$ . Conversely if the infimum on a homotopy class of a functional of at least conformal degree vanishes then the map is trivial in homology of high degrees.*

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## 1 Introduction

The main objective of this paper is a correspondence between universal energy functionals on one side and factorization over subskeleta up to homotopy on the other. By an energy functional  $E_\phi: C^\infty(M, V) \rightarrow \mathbb{R}^+$  we mean an integral  $\int_M \phi(\nabla f, \dots, \nabla^r f)$  of a constant coefficient differential operator  $\phi$  over  $M$ , where  $M$  and  $V$  are smooth compact connected Riemannian manifolds, see section 2.

By a theorem of White, [6], the infimum

$$\tilde{E}_\phi(f) := \inf \{E_\phi(g) | g: M \rightarrow V, g \simeq f\}$$

of a first order functional depends only on the homotopy class of the restriction of  $f$  to the  $d$ -skeleton  $M^d$  of a triangulation of  $M$ , where  $d$  is the degree of  $\phi$ . Theorem 3.1 contains an analogous statement for functionals involving higher derivatives of  $f$ .

For first order functionals we consider a kind of converse, namely the implications of  $\tilde{E}_\phi(f) = 0$ , restricting our considerations to functionals of (at least) conformal degree  $d = n = \dim M$ . For instance if  $\phi(df) = \det(df^*df)^{1/2}$  and  $\tilde{E}_\phi(f) = 0$  then  $f$  is homotopic to a map  $\tilde{f}: M \rightarrow V^{n-1}$  into the  $n - 1$ -skeleton of

a triangulation of  $V$ . For the  $n$ -norm  $\phi(df) = \|df\|^n = \text{Tr}((df^*df)^{n/2})$  it is proved in [6] that  $\tilde{E}_\phi(f) = 0$  implies that  $f \simeq * = V^0$  is nullhomotopic.

Theorem 3.2 interpolates homologically between these facts. Since  $\phi$  depends explicitly only on the differential  $df$ , the functional  $E_\phi$  corresponds biuniquely to a symmetric function in  $n$  variables, see section 2. Fundamental are the Jacobians, i.e. the conformal powers  $\phi(df) = \sigma_l^{n/2l}(df^*df)$  of the elementary symmetric polynomials  $\sigma_l$  in the eigenvalues of  $df^*df$ . Proposition 2.1 establishes a hierarchy  $\sigma_1^{n/2} \gg \sigma_2^{n/4} \gg \dots \sigma_l^{n/2l} \gg \dots \gg \sigma_n^{1/2}$ , where  $\phi \gg \psi$  indicates that we can estimate the energies  $CE_\phi(f) \geq E_\psi(f)$  with a constant  $C$  independent of  $f$ . With respect to  $\gg$  the  $n$ -norm  $\|df\|^n = \text{Tr}((df^*df)^{n/2})$  is equivalent to  $\sigma_1^{n/2}(df^*df) = (\text{Tr}(df^*df))^{n/2}$ . If  $\phi \gg \sigma_l^{n/2l}$  then  $\tilde{E}_\phi(f) = 0$  implies that  $f$  induces 0 in (co)homology of degree  $\geq l$  with arbitrary coefficients, just like a map which factors over the  $(l-1)$ -skeleton  $V^{l-1} \hookrightarrow V$  of some triangulation of  $V^0 \subset V^1 \subset \dots \subset V^{k-1} \subset V$ .

If  $f$  is homotopic to a map  $\tilde{f}: M \rightarrow V^{l-1} \subset V$  then clearly  $\tilde{E}_{\sigma_l^r}(f) = 0$  for all  $r$ . Actually we have  $\tilde{E}_\Phi(f) = 0$  for all polynomials  $\Phi$  of length  $\geq l$ . I do not know in how far a converse holds. Theorem 3.2 only yields the vanishing of the primary obstructions in this factorization problem and this needs at least conformal degree  $r \geq n/2l$ . Proposition 3.3 shows that a general converse requires to assume even higher than conformal degree.

Rationally these obstructions can be expressed by integrals over  $M$  of certain forms constructed from minimal models, as in [2],[5] for instance. The energy functionals behave like an absolute value thereof, similar to the case of mapping degree and volume for maps between manifolds of the same dimension  $n = k$ . Consider as an example the 2-Jacobian  $\phi(A) = \sigma_2^{3/4}(A^*A)$  for maps  $M^3 \rightarrow V^2$ , i.e.  $E_\phi(f) = \int_M |f^*\omega|^{3/2}$  where  $\omega$  is the volume form of  $V$ . In [4] it is shown that the Hopf invariant of a map  $M = S^3 \rightarrow S^2 = V$  can be estimated by  $E_{\sigma_2^{3/4}}(f)$ . Thus if  $\tilde{E}_{\sigma_2^{3/4}}(f) = 0$  then  $f$  is nullhomotopic. More generally the infinite higher homotopy groups of spheres are  $\pi_{4k-1}(S^{2k}) = \mathbb{Z} \oplus G$ ,  $G$  finite, and the Hopf invariant  $h_k: \pi_{4k-1}(S^{2k}) \rightarrow \mathbb{Z}$  detects the free part. As before one gets that  $h_k(f) = 0$  if  $\tilde{E}_{\sigma_{2k}^{(4k-1)/2k}}(f) = 0$  but if  $k > 1$  this does not imply that  $f$  is nullhomotopic. I am grateful to T. Rivière for this remark.

## 2 Energy functionals

For smooth maps  $f: M^n \rightarrow V^k$  of compact Riemannian manifolds we consider functionals

$$E_\phi(f) := \int_M \phi(\nabla f, \nabla^2 f, \dots, \nabla^r f),$$

of order  $r$  parametrized by functions

$$\phi : \bigoplus_{j=1}^r \left( (\mathbb{R}^n)^{\otimes j} \otimes \mathbb{R}^k \right) \rightarrow \mathbb{R}_0^+ .$$

These functions are assumed invariant under the (diagonal) action of  $O(n) \times O(k)$ . They can therefore be evaluated on the derivatives  $\nabla^j f \in \Gamma(TM^{*\otimes j} \otimes f^*TV)$  to yield well defined energy densities  $\phi(\nabla f, \nabla^2 f, \dots, \nabla^r f)$ . We will say that  $\phi$  (or  $E_\phi$ ) has degree  $\leq d$  if it satisfies an estimate

$$|\phi(A_1, \dots, A_r)| \leq C(1 + \|A_1\|^d + \|A_2\|^{d/2} + \dots + \|A_r\|^{d/r}) , \quad A_j \in (\mathbb{R}^n)^{\otimes j} \otimes \mathbb{R}^k .$$

Denoting by  $df^*$  the adjoint of  $df$  and using the Riemannian metrics to identify the dual tangent bundles we get a bundle endomorphism  $df^*df$  of  $TM$ . Examples of functionals as above are the classical (2-)energy, or more general  $p$ -energy  $\int_M (\text{Tr}(df^*df))^{p/2}$  of degree  $p$ , the exponential energy  $\int_M e^{\text{Tr}(df^*df)}$  of infinite degree, the Willmore energies  $W(f) := \int_M \|\nabla df\|^2$  of degree 4,  $W_0(f) := \int_M \|\nabla df(df^*df)^{-1}\|^2$ . For an immersion  $W_0$  is the  $L^2$ -norm of the mean curvature.

Functionals of first order, i.e. involving only first order derivatives, are conveniently described by nonnegative functions  $\Phi: M(n \times n, \mathbb{R})^+ \rightarrow \mathbb{R}_0^+$  defined on nonnegative symmetric matrices and invariant under conjugation by  $O(n)$ . We have  $\Phi(A^*A) = \phi(A)$  and

$$E_\phi(f) = E_\Phi(f) := \int_M \Phi(df^*df) d\text{vol}_g .$$

We describe the estimates between polynomial energy functionals corresponding to the successive lifting problem for a map  $f: M \rightarrow V$  over the skeleta of  $V$ .

An invariant function  $\Phi$  on symmetric matrices is essentially the same as a symmetric function in the eigenvalues. For an arbitrary symmetric function  $\Phi$  in  $n$  variables denote by

$$l(\Phi) := \min\{r \mid \Phi(x_1, \dots, x_r, x_{r+1} = 0, \dots, x_n = 0) \neq 0\} .$$

the length of  $\Phi$ . If  $\Phi$  is polynomial (eventually with real exponents) then  $l(\Phi)$  is the length of the shortest monomial occurring in  $\Phi$ . By  $((x_1^{i_1} \cdots x_s^{i_s}))$ , for  $i_1 \geq i_2 \geq \dots \geq i_s$ ,  $i_j \in \mathbb{R}_0^+$ , we denote the symmetrization of  $x_1^{i_1} \cdots x_s^{i_s}$ , i.e. the function

$$\sum_{\pi \in S_n} x_{\pi 1}^{i_1} \cdots x_{\pi s}^{i_s} .$$

We have a simple hierarchy of the functionals  $E_{((x_1^{i_1} \cdots x_s^{i_s}))}$ :

**Proposition 2.1** *1. If  $1 \leq r < s \leq l \leq n$  and  $i_s \geq \epsilon \geq 0$  then*

$$E_{((x_1^{i_1} \cdots x_l^{i_l}))}(f) \leq n! E_{((x_1^{i_1} \cdots x_r^{i_r + \epsilon} \cdots x_s^{i_s - \epsilon} \cdots x_l^{i_l}))}(f) .$$

2. For any  $\mu > 0$ ,  $r \geq 1$  we have

$$E_\Phi \leq \frac{1}{\mu^{r-1}} E_{\Phi^r} + \mu \operatorname{vol} M .$$

**Proof:** Let  $a_1(x) \geq a_2(x) \geq \dots \geq a_n(x)$  be the eigenvalues of  $d_x f^* d_x f$ ,  $x \in M$ . Now

$$\begin{aligned} E_{((x_1^{i_1} \dots x_l^{i_l}))}(f) &= \int_M \sum_{\pi \in S_n} a_{\pi 1}^{i_1} \dots a_{\pi r}^{i_r} \dots a_{\pi s}^{i_s} \dots a_{\pi l}^{i_l} \\ &\leq n! \sum_{\pi \in S_n} \int_{a_{\pi 1} \geq \dots \geq a_{\pi l}} a_{\pi 1}^{i_1} \dots a_{\pi r}^{i_r} \dots a_{\pi s}^{i_s} \dots a_{\pi l}^{i_l} \\ &\leq n! \sum_{\pi \in S_n} \int_{a_{\pi 1} \geq \dots \geq a_{\pi l}} a_{\pi 1}^{i_1} \dots a_{\pi r}^{i_r + \epsilon} \dots a_{\pi s}^{i_s - \epsilon} \dots a_{\pi l}^{i_l} \\ &\leq n! E_{((x_1^{i_1} \dots x_r^{i_r + \epsilon} \dots x_s^{i_s - \epsilon} \dots x_l^{i_l}))}(f) \end{aligned}$$

For the second claim we decompose the domain of integration to estimate

$$\begin{aligned} \int_M \Phi(d_f^* d_f) &\leq \int_{\{x | \Phi(d_x f^* d_x f) > \mu\}} \frac{\Phi(d_f^* d_f)^r}{\mu^{r-1}} + \int_{\{x | \Phi(d_x f^* d_x f) \leq \mu\}} \mu \\ &\leq \frac{1}{\mu^{r-1}} E_{\Phi^r} + \mu \operatorname{vol} M . \end{aligned}$$

•

We denote by  $\sigma_l(A)$  the  $l$ th elementary symmetric polynomial in the eigenvalues of a matrix  $A$ . It is determined by the identity  $\det(1 + \lambda A) = \sum_l \sigma_l(A) \lambda^l$ . If  $\Phi$  and  $\Psi$  are homogeneous polynomials with  $l(\Psi) \leq l(\Phi) = l$  then there are constants  $\tilde{C} = \tilde{C}(\Psi, \Phi, n)$  and  $C = C(\Phi, n, l)$  such that

$$\tilde{C} C E_\Psi(f) \geq C E_\Phi(f) \geq E_{\sigma_l^{n/2l}}(f)$$

for all  $f$ .

### 3 Statement of Results

**Theorem 3.1** *Let  $\phi$  have degree  $d$  and let  $L \subset M$  be a submanifold of codimension  $q$  with  $q > d$ . If  $f, g: M \rightarrow V$  are homotopic on  $M \setminus L$  then  $\tilde{E}_\phi(f) = \tilde{E}_\phi(g)$ .*

For instance in order to compute  $\tilde{W}(f)$  we may modify  $f$  in a tubular neighbourhood of a submanifold  $L$  of codimension at least 5. Even more specifically  $\tilde{W}(f) = 0$  for any map  $S^n \rightarrow V$  if  $n \geq 5$ . It was proved by B. White in [6], see also Pluzhnikov [3], that for functionals of order 1 the same holds if  $f$  and  $g$  are homotopic on a  $(q-1)$ -skeleton of  $M$  with  $q > d$ , i.e. outside a skeleton of codimension  $q > d$ .

**Theorem 3.2** Assume that  $\phi(A) \geq \sigma_l(A^*A)^{n/2l}$  and that  $\tilde{E}_\phi(f) = 0$ . Then  $f$  induces 0 in (co)homology groups of degree  $\geq l$ .

The following proposition shows that the assumptions of theorem 3.2 do not in general imply that  $f$  is homotopic to a map to  $\tilde{f}: M \rightarrow V^{l-1} \subset V$ .

**Proposition 3.3** Let  $n = m + m'$ ,  $k = l + l'$ ,  $g: S^m \rightarrow S^l$ ,  $g': S^{m'} \rightarrow S^{l'}$  and  $f = g \times g': S^m \times S^{m'} \rightarrow S^l \times S^{l'}$ . Assume that  $r < m/2l$  or  $r < m'/2l'$ . Then  $\tilde{E}_\phi(g \times g') = 0 = \tilde{E}_\phi(g \wedge g')$  for  $\phi = \sigma_k^r$  where  $g \wedge g': S^{m+m'} = S^m \wedge S^{m'} \rightarrow S^{l+l'}$  denotes the smash product of  $g$  and  $g'$ . In particular if  $g: S^{n-1} \rightarrow S^{k-1}$  and  $f = Sg = g \wedge id: S^m \rightarrow S^k$  is the suspension of  $g$  then  $\tilde{E}_\phi(f) = 0$  for  $\phi = \sigma_k^r$  if  $r < (n-1)/2(k-1)$ .

**Proof:** For the cartesian product  $f = g \times g'$  we get

$$df^*df = \begin{pmatrix} dg^*dg & 0 \\ 0 & dg'^*dg' \end{pmatrix},$$

$$\sigma_k(df^*df) = \sigma_l(dg^*dg)\sigma_{l'}(dg'^*dg')$$

and

$$\tilde{E}_\phi(f) = \tilde{E}_{\sigma_l^r}(g)\tilde{E}_{\sigma_{l'}^r}(g').$$

If  $r < m/2l$  or  $r < m'/2l'$  then one of the factors vanishes by [6] or theorem 3.1, hence  $\tilde{E}_{\sigma_k^r}(f) = 0$ .

The assertion about the smash product follows from that for the cartesian product. We deform  $g$  and  $g'$  to maps which are constant on some open sets  $U \subset S^m$  and  $U' \subset S^{m'}$  containing the base points. The projection  $\pi: S^m \times S^{m'} \rightarrow S^m \wedge S^{m'}$  and its inverse have bounded differential over  $S^m \setminus U \times S^{m'} \setminus U'$ . Hence we can estimate

$$\begin{aligned} E_\phi(g \wedge g') &= \int_{\pi(S^m \setminus U \times S^{m'} \setminus U')} \sigma_k^r(g \wedge g') \leq C \int_{S^m \setminus U \times S^{m'} \setminus U'} \sigma_k^r(\pi' \circ (g \times g')) \\ &\leq C \int_{S^m \times S^{m'}} \sigma_k^r(g \times g') = CE_{\sigma_l^r}(g \times g') \end{aligned}$$

where  $\pi': S^l \times S^{l'} \rightarrow S^l \wedge S^{l'}$ . •

The generator of  $\pi_4(S^3) \cong \mathbb{Z}/2$  is the suspension of the Hopf map  $\eta: S^3 \rightarrow S^2$ . Hence  $\tilde{E}_{\sigma_3^r}(s\eta) = 0$  for  $r < 3/4$  but the conformal power is  $r = 2/3$ .

## 4 Proof of Theorem 3.1

Let  $R$  be small enough such that the exponential map restricted to the  $R$ -disc bundle of the normal bundle of  $L$  is a diffeomorphism, i.e. the map

$$\begin{aligned} RD\nu(L) &\rightarrow U_R(L) := \{x \in M \mid d(x, L) \leq R\} \\ v \in \nu_p(L) &\mapsto \exp_p(v) . \end{aligned}$$

We will write  $\rho(x) = d(x, L)$  for the distance from  $L$  and  $\mu(r, x) = \exp_p(rv)$  for  $x = \exp_p(v) \in U_R(L)$  and  $r \leq R/\rho(x)$ . Let  $0 < \alpha < \beta = 2\alpha < \epsilon = 3\alpha < R$  and choose smooth functions  $\chi: [0, R] \rightarrow [0, 1]$  and  $\psi: [0, R] \rightarrow [0, R/\alpha]$  such that  $\chi(x) = 0$  for  $x \in [0, \alpha]$ ,  $\chi(x) = 1$  for  $x \in [\beta, R]$ ,  $\chi' \geq 0$  and  $\psi(x) = R/\alpha$  for  $x \in [0, \alpha]$ ,  $\psi(x) = 1$  for  $x \in [\epsilon, R]$ ,  $\psi' \leq 0$  and  $R/x \geq \psi(x) \geq R/2x$  for  $x \in [\alpha, \beta]$ . If the functional in question has order  $r$  we also require that for the derivatives  $\chi^{(j)}$  and  $\psi^{(j)}$ ,  $j = 0 \dots r$ , we have that  $|\chi^{(j)}(x)| \leq C/\alpha^j$  and  $|\psi^{(j)}| \leq C/\alpha^{j+1}$ . Here and in the sequel  $C$  denotes a suitable constant independent of  $\alpha$ . Denote by  $\pi$  the map  $M \rightarrow M \times [0, 1]$  given by

$$\pi(x) := (\mu(\psi(\rho(x)), x), \chi(\rho(x))) .$$

Note that the image of  $\pi$  is contained in the compact set

$$K := M \times \{1\} \cup M \setminus U_{R/2}(L) \times [0, 1] \cup M \times \{0\} .$$

Let  $H: K \rightarrow V$  be a smooth homotopy between  $f = H|_{M \times \{0\}}$  and  $g|_{M \times \{1\}}$  outside  $L$ . The map  $h$  defined by  $h(x) := H(\pi(x))$  is homotopic to  $f$  and coincides with  $g$  on  $M \setminus U_\epsilon(L)$ . Thus

$$\tilde{E}_\phi(f) - \tilde{E}_\phi(g) = \tilde{E}_\phi(h) - \tilde{E}_\phi(g) = \int_{U_\epsilon} \phi(\nabla h, \dots, \nabla^r h) - \int_{U_\epsilon} \phi(\nabla g, \dots, \nabla^r g) .$$

For the proof of the theorem it suffices to show that the energy of  $h$  on  $U_\epsilon(L)$  can be made arbitrarily small by choosing  $\alpha$  appropriately. So we need to estimate the (covariant) derivatives of order  $\leq r$  of  $h$ . Up to some constants depending on the derivatives of  $H$  on the compact set  $K$  we have to estimate  $\nabla^j \pi$ ,  $j = 1 \dots r$ . At  $p \in L$  we use a chart  $\xi$  of  $U_R(L)$  mapping  $L$  to  $\mathbb{R}^{n-q} \times \{0\}$ , preserving  $\rho$ , i.e.  $\xi(z) = (x, y)$  with  $\rho(z) = |x|$ , and such that  $\xi(\mu(r, z)) = (rx, y)$ . Up to constants depending on derivatives of  $\xi$  we have to estimate the derivatives of

$$(x, y) \mapsto (\psi(|x|x), y, \chi(|x|)) .$$

A straightforward calculation shows that

$$\begin{aligned} \nabla^j \chi(|x|) &= \sum_{i=1}^j \chi^{(i)}(|x|) \mathcal{O}(1/|x|^{j-i}) , \\ \nabla^j \psi(|x|x) &= \sum_{i=1}^j \psi^{(i)}(|x|) \mathcal{O}(1/|x|^{j-i-1}) . \end{aligned}$$

Hence

$$\|\nabla^j \chi(|x|)\| \leq C\alpha^{-j} \quad \text{and} \quad \|\nabla^j (\psi(|x|x))\| \leq C\alpha^{-j}.$$

Therefore we can estimate  $\nabla^j \pi$  by  $\alpha^{-j}$  and the energy integral by

$$\begin{aligned} E_\phi(h|_{U_\epsilon(L)}) &= \int_{U_\epsilon(L)} \phi(\nabla h, \dots, \nabla^r h) \\ &\leq \text{vol} U_\epsilon(L) C(1 + |\nabla h|^d + \dots + |\nabla^r h|^{d/r}) \\ &\leq C\epsilon^q \alpha^{-d} = C3^q \alpha^{q-d} \end{aligned}$$

If  $q > d$  then this tends to 0 for  $\alpha \rightarrow 0$ . •

## 5 Proof of Theorem 3.2

Let  $M^q, V^q$  denote the  $q$ -skeleta of smooth triangulations of  $M$  and  $V$  respectively. By the Deformation Theorem of geometric measure theory, [1], we can deform the restriction  $f|_{M^q}$  of  $f$  to a Lipschitz map  $\tilde{f}$  mapping  $M^q$  to  $V^q$  such that for the  $q$ -dimensional Hausdorff measures  $\mathcal{H}$  we have

$$\mathcal{H}^q(\tilde{f}(M^q)) \leq C_1 \mathcal{H}^q(f(M^q))$$

where  $C_1$  does not depend on  $f$ .

It suffices to consider the case  $\Phi = \sigma_l^{n/2l}$ . We choose an embedding of the tangent bundle  $TM \hookrightarrow M \times \mathbb{R}^s$  into a trivial bundle of rank  $s$ . Let  $\pi$  be the orthogonal projection  $M \times \mathbb{R}^s \rightarrow TM$ . Chose  $\epsilon$  small enough so that  $\Psi_v: m \mapsto \exp_m(\pi v)$  is a diffeomorphism of  $M$  for all  $v \in D^s = \{v \in \mathbb{R}^s \mid |v| \leq \epsilon\}$ . Also let  $U_\epsilon M^q$  denote the image of  $M^q \times D^s$  under the map  $\Psi: (m, v) \mapsto \exp_m(\pi v)$ . Let  $f_v(m) = f(\exp_m(\pi v))$ . We denote by  $e_\Phi(f) = \Phi(df^*df)$  the energy density and apply the transformation formula to the submersion  $\Psi: M^q \times D^s \rightarrow U_\epsilon M^q$ . Let  $J_n \Psi$  be the  $n$ -Jacobian of this map,

$$C_2 = \frac{1}{\max_{z \in U_\epsilon M^q} \mathcal{H}^{s+q-n}(\Psi^{-1}(z))}$$

and

$$C_3 = \frac{\min_{M^q \times D^s} J_n \Psi}{\max_{z \in U_\epsilon M^q} \mathcal{H}^{s+q-n}(\Psi^{-1}(z)) \max_{m,v} |d_m \psi_v^* d_m \psi_v|^{n/2}},$$

where  $|d_m \psi_v^* d_m \psi_v|$  denotes the operator norm of  $d_m \psi_v^* d_m \psi_v$  for  $m \in M^q, v \in D^s$ , i.e. its largest eigenvalue. Then

$$\begin{aligned} E_\Phi(f) &\geq \int_{U_\epsilon M^q} e_\Phi(f) \\ &\geq C_2 \int_{U_\epsilon M^q} e_\Phi(f) \mathcal{H}^{s+q-n}(\Psi^{-1}(z)) dz \end{aligned}$$

$$\begin{aligned}
&= C_2 \int_{M^q \times D^s} e_{\Phi}(f)(\Psi(y)) J_n \Psi(y) dy \\
&\geq C_2 \min_{M^q \times D^s} J_n \Psi \int_{M^q \times D^s} e_{\Phi}(f)(\Psi(y)) dy \\
&= C_2 \min_{M^q \times D^s} J_n \Psi \int_{D^s} \int_{M^q} e_{\Phi}(f)(\Psi_v(m)) dm dv \\
&\geq C_3 \int_{D^s} \int_{M^q} e_{\Phi}(f \circ \Psi_v) dm dv . \tag{5.1}
\end{aligned}$$

Assume that  $E_{\Phi}(f) \leq \delta$  for some  $\delta$ . By (5.1) we find  $v \in D^s$  such that

$$\int_{M^q} e_{\Phi}(f_v|_{M^q})(m) dm \leq \delta/C_3 .$$

From proposition 2.1 there is a constant  $C_4$  such that

$$\begin{aligned}
\mathcal{H}^q(f_v(M^q)) &= E_{\sigma_q^{1/2}} \leq C_4 E_{\sigma_l^{q/2l}} \leq C_4 \left( \frac{1}{\mu^{n/q-1}} E_{\sigma_l^{n/2l}} + \mu \text{vol} M \right) \\
&\leq C_4 \left( \frac{1}{\mu^{n/q-1}} \delta/C_3 + \mu \text{vol} M \right)
\end{aligned}$$

for any  $\mu > 0$  provided  $q \geq l$ . Choosing  $\delta$  and  $\mu$  appropriately we can achieve that this is strictly less than the volume of the smallest  $q$ -cell of  $V^q$ . Thus we get a map  $\tilde{f}_v$  homotopic to  $f$  which maps  $M^q$  to  $V^q$  and misses a point in every  $q$ -cell of  $V^q$ . Therefore it is homotopic to a map  $M^q \rightarrow V^{q-1}$ .  $\bullet$

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