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On the Calculation of the $l_2 \to l_1$ Induced Matrix Norm

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Abstract

We show that the $l_2 \to l_1$ induced matrix norm, namely the norm induced by the l_2 and l_1 vector norms in the domain and range space, respectively, can be calculated as the maximal element of a finite set involving discrete additive combinations of the rows of the involved matrix with weights of ± 1; the number of elements this set contains is exponential in the number of rows involved. A geometric interpretation of the result allows us to extend the result to some other induced norms. Finally, we generalize the findings to bounded linear operators on separable Banach spaces that can be obtained as strong limits of sequences of finite-dimensional linear operators.

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1 Introduction

Induced matrix norms are routinely used in both pure and applied mathematics. In many applications only an estimate or bound on norm is necessary, and extensive results have been obtained on the estimation or bounding of various

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matrix norms [Higham(1992), Horn and Johnson(1985)], but few results exist on the exact calculation of such norms. There is growing interest in exact expressions for induced norms involving the l_p-norm for $p \leq 1$, as they are arise in sparse representations, coding, signal processing, and compressive sampling [Chen et al.(1998)Chen, Donoho, and Saunders], [Donoho and Elad(2003)], [Donoho(2006)], [Zibulevsky and Pearlmutter(2001)], [Olshausen and Field(2004)], [Lewicki and Sejnowski(2000)], [Lewicki(2002)], [Smith and Lewicki(2006)], [Candès and Tao(2006)].

We derive a simple expression for the $l_2 \to l_1$ induced matrix norm, and generalize it to infinite-dimensional matrices that play the role of bounded linear transformations between separable Banach spaces. We also derive a simple expression for the $l_p \to l_q$ induced matrix norm when $0 < p \leq 1 \leq q$.

The usual l_p norms[2] over vectors, $\|\mathbf{x}\|_p \equiv \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for $0 < p < \infty$ and $\|\mathbf{x}\|_\infty \equiv \max_{i\in\{1,\ldots,n\}}|x_i|$, can be used to induce a norm on mappings between two normed spaces: the maximum ratio between the norm of a vector in the domain and the norm of the corresponding vector in the domain [Horn and Johnson(1985), pp. 257–335]. We consider a matrix as a linear mapping from one normed linear space to another, so the $\|\cdot\|_{pq}$-norm of a matrix $A : \mathbb{R}^{m\times n}$ is defined to be the maximal ratio of the l_q-norm of the image of a vector in the range space over the l_p-norm of that vector in the domain space. Because $\|c\mathbf{x}\|_r = c\|\mathbf{x}\|_r$ for $c > 0$, we can constrain the pre- or post-image vector to unit length without loss of generality.

$$\|A\|_{pq} \equiv \sup_{\mathbf{x}\in\mathbb{R}^n-\{\mathbf{0}\}} \frac{\|A\mathbf{x}\|_q}{\|\mathbf{x}\|_p} = \sup_{\{\mathbf{x}\in\mathbb{R}^n:\|\mathbf{x}\|_p=1\}} \|A\mathbf{x}\|_q = \left(\min_{\{\mathbf{x}\in\mathbb{R}^n:\|A\mathbf{x}\|_q=1\}} \|\mathbf{x}\|_p\right)^{-1} \quad (1)$$

The supremum here is a maximum, as the spaces considered are finite-dimensional and the unit l_p-sphere is bounded and closed. As shorthand, we define $\|A\|_p \equiv \|A\|_{pp}$.

Closed-form solutions are known for some induced norms, in particular cases involving the l_1, l_2, and l_∞-norms [Higham(1992)]. We now list some $l_p \to l_q$ induced matrix norms and their closed-form exact solutions. These are listed in lexicographic order by p and q. Novel expressions, derived below, are starred.

$$\|A\|_{pq} = \max_{j=1,\ldots,n} \|\mathbf{a}^j\|_q \qquad (p \leq 1 \leq q) \quad (2a^*)$$

$$\|A\|_1 = \max_{j=1,\ldots,n} \|\mathbf{a}^j\|_1 = \|A^T\|_\infty \quad (2b)$$

$$\|A\|_{12} = \max_{j=1,\ldots,n} \|\mathbf{a}^j\|_2 = \|A^T\|_{\infty 2} \quad (2c)$$

[2] Although when $p < 1$ the triangle inequality does not hold, this does not affect us below. For clarity we write "l_p norm" instead of "l_p seminorm" or "l_p norm-or-possibly-seminorm", and similarly we write "induced norm" instead of "induced norm-or-possibly-seminorm."

$$\|A\|_{21} = \max_{\mathbf{s}\in\{-1,+1\}^m} \left\|\mathbf{s}^T A\right\|_2 \tag{2d*}$$

$$\|A\|_2 = \max_{\lambda\in\mathrm{eig}(A^T A)} \sqrt{|\lambda|} = \|A^T\|_2 \tag{2e}$$

$$\|A\|_{\infty 1} = \max_{\mathbf{s}\in\{-1,+1\}^m} \|A\mathbf{s}\|_1 \tag{2f}$$

$$\|A\|_{1\infty} = \max_{(i,j)\in\{1,\dots,m\}\times\{1,\dots,n\}} |a_{ij}| \tag{2g}$$

$$\|A\|_{\infty 2} = \max_{i=1,\dots,m} \|\mathbf{a}_i\|_2 = \|A^T\|_{12} \tag{2h}$$

$$\|A\|_{\infty} = \max_{i=1,\dots,m} \|\mathbf{a}_i\|_1 = \|A^T\|_1 \tag{2i}$$

where $\mathrm{eig}\, X$ is the set of eigenvalues of the array X, the vector $\mathbf{a}^j$ is the j-th column of A, and the vector $\mathbf{a}_i$ is the i-th row of A.

Out of these, (2b), (2e), and (2i) are well known and can be found in most textbooks of matrix theory or numerical analysis; (2f) and (2g) appear in [Rohn(2000)]; (2c) and (2h) appear in [Higham(1992)] (the former is a special case of the more general novel expression (2a*)). We now proceed to prove the novel cases, (2a*) in Section 5, and (2d*) in Section 2, using a combination of algebraic and geometric arguments. Section 6 generalizes some of the results to linear operators on Banach spaces.

2 $\|\cdot\|_{21}$

We first show that the $\|A\|_{21}$ induced matrix norm can be computed by considering a maximum over a finite set of vectors.

Theorem 1. *Let $A : \mathbb{R}^{n\times m}$, and let $\mathbf{a}_i$, $i = 1, \dots, m$ be the row vectors of A; then,*

$$\|A\|_{21} = \max_{\mathbf{s}\in\{-1,+1\}^m} \left\|\mathbf{s}^T A\right\|_2 = \max_{\mathbf{s}\in\{-1,+1\}^m} \left\|\sum_{i=1}^{n} s_i \mathbf{a}_i\right\|_2 \tag{2d*}$$

Proof. We need the maximum of $\|\mathbf{Ax}\|_1$ subject to the constraint that $\|\mathbf{x}\|_2 = 1$. Observe that the constraint is continuously differentiable, while the function to be maximized is continuous but only piecewise differentiable. Within a differentiability region, we may use Lagrange multipliers to maximize; over regions of non-differentiability, however, we need to find the maximum manually. Fortunately, we can prove that the global maximum has to lie within a differentiability region (see Figure 1), so this last step is not necessary.

First we compute the local maxima within each differentiability region using Lagrange multipliers. Consider the function

$$f(\mathbf{x}, \lambda) \equiv \|A\mathbf{x}\|_1 + \lambda(\|\mathbf{x}\|_2 - 1) = \sum_{i=1}^{m} \left|\sum_{j=1}^{n} a_{ij} x_j\right| + \lambda\left(\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} - 1\right).$$

We need to find the partial derivatives of f with respect to x_i and λ and set them to zero, but the absolute values in the formula make this difficult. We can circumvent this by setting $s_i = \operatorname{sign}(A\mathbf{x})_i$.

$$f(\mathbf{x}, \lambda) = \sum_{i=1}^{m} s_i \sum_{j=1}^{n} a_{ij} x_j + \lambda\left(\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} - 1\right)$$

This function is now clearly differentiable in the set

$$W = \mathbb{R}^n - \bigcup_{i=1}^{m} \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i \mathbf{x} = 0\},$$

which is a collection of connected regions, separated by hyperplanes, within each of which f is differentiable and the components of $\mathbf{s}$ are constant. It follows that, in W,

$$\frac{\partial f}{\partial x_k} = \sum_{i=1}^{m} s_i a_{ik} + \lambda \frac{x_k}{\|x\|_2} = \sum_{i=1}^{m} s_i a_{ik} + \lambda x_k = 0,$$

so $\mathbf{x} = -\frac{1}{\lambda} A^T \mathbf{s}$, and therefore $|\lambda| = \|A^T \mathbf{s}\|_2$ and

$$f(\mathbf{x}, \lambda) = -\frac{1}{\lambda} \sum_{i=1}^{m} s_i \sum_{k=1}^{n} a_{ik} \sum_{j=1}^{m} s_j a_{jk} = -\frac{1}{\lambda} \sum_{k=1}^{n} \left(\sum_{i=1}^{m} s_i a_{ik}\right)\left(\sum_{j=1}^{n} s_j a_{jk}\right) = \frac{\lambda^2}{\lambda} = \lambda.$$

Observe further that $\displaystyle\sum_{i=1}^{m} s_i \mathbf{a}_i = \mathbf{s}^T A$ so $\displaystyle\left\|\sum_{i=1}^{m} s_i \mathbf{a}_i\right\|_2 = \sqrt{\mathbf{s}^T A A^T \mathbf{s}}$.

What about the values of f outside W? Can perhaps the maximum be found there? We argue that this is not possible. Let $A : \mathbb{R}^{m \times n}$ be an array for which the vector $\mathbf{x}$ yielding the maximum satisfies, without loss of generality, the $l \leq m$ conditions $\mathbf{a}_i \mathbf{x} = 0$, $i = 1, \ldots, l$. Now form the vector $\mathbf{y} = \dfrac{\mathbf{x} + \epsilon \mathbf{a}_1}{\sqrt{1 + \epsilon^2 |\mathbf{a}_1|^2}}$, and assume that ϵ is small enough so that $s_i = \operatorname{sign}(\mathbf{a}_i \mathbf{y})$, $i = l+1, \ldots, m$. Then,

$$\begin{aligned}
\sum_{i=1}^{m} |\mathbf{a}_i \mathbf{y}| &= \frac{1}{\sqrt{1 + \epsilon^2 |\mathbf{a}_1|^2}} \left[\sum_{i=l+1}^{m} |\mathbf{a}_i \mathbf{x}| + |\epsilon| \sum_{i=1}^{l} |\mathbf{a}_i \mathbf{a}_1| + \epsilon \sum_{i=l+1}^{m} s_i \mathbf{a}_i \mathbf{a}_1\right] \\
&= \sum_{i=l+1}^{m} |\mathbf{a}_i \mathbf{x}| + |\epsilon| \sum_{i=1}^{l} |\mathbf{a}_i \mathbf{a}_1| + \epsilon \sum_{i=l+1}^{m} s_i \mathbf{a}_i \mathbf{a}_1 + O(\epsilon^2) \\
&= \sum_{i=l+1}^{m} |\mathbf{a}_i \mathbf{x}| + C_1 |\epsilon| + C_2 \epsilon + O(\epsilon^2)
\end{aligned}$$

Assuming $\mathbf{a}_1 \neq 0$, we get $C_1 > 0$, while regardless of the sign of C_2 we can choose the sign of ϵ appropriately so that $C_2\epsilon \geq 0$. This makes $C_1|\epsilon| + C_2\epsilon > 0$, and

$$\sum_{i=1}^{m} |\mathbf{a}_i \mathbf{y}| > \sum_{i=l+1}^{m} |\mathbf{a}_i \mathbf{x}|$$

contradicting the assumption that the maximum occurs at $\mathbf{x}$.

The requirement that $\mathbf{a}_1 \neq 0$ can be eased by the continuity of vector norms, which allows us to approximate A by a sequence of arrays $\{A_k\}_{k\in\mathbb{N}}$ with non-zero rows. This completes the proof. □

As a final observation, we note that the maxima occur in pairs: if $\mathbf{s}$ is a maximizing vector then $-\mathbf{s}$ is as well.

3 Geometric intuition

Let us now develop a geometric intuition for the $l_2 \to l_1$ induced norm of A. We wish to find the smallest $t > 0$ such that

$$\exists \mathbf{x} : \|\mathbf{x}\|_2 = t \text{ given that } \|A\mathbf{x}\|_1 = 1.$$

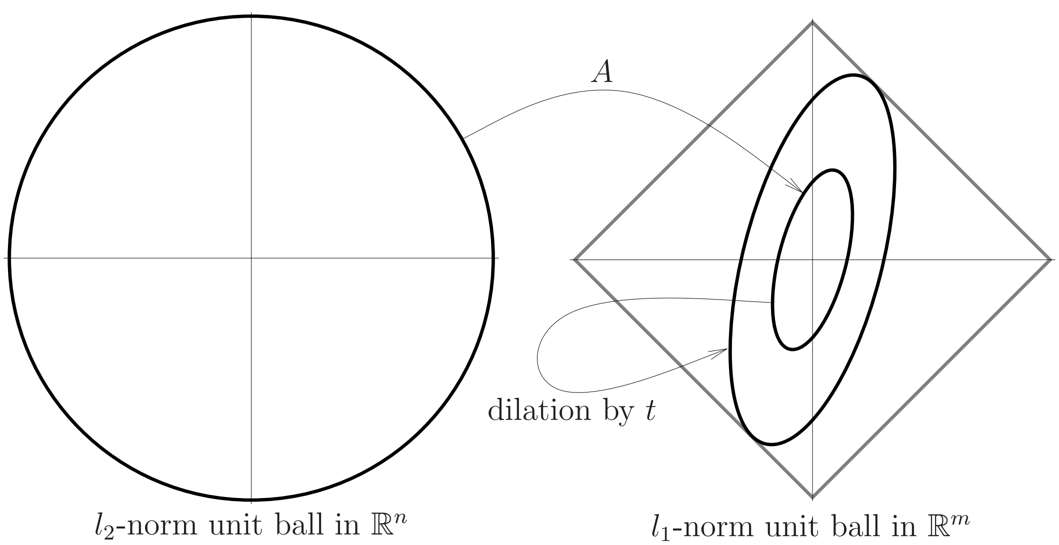

Figure 1: Geometric intuition of $\|\cdot\|_{21}$ induced matrix norm. The number $t = \|A\|_{21}$ is the smallest number for which $\|\mathbf{y}\|_1 = 1$ where $\mathbf{y} = tA\mathbf{x}$ and $\|\mathbf{x}\|_2 = 1$. This means we can consider the image of the l_2-norm unit ball in $\mathbb{R}^n$ under A, namely $\{\mathbf{y} : \mathbf{y} = A\mathbf{x}, \|\mathbf{x}\|_2 = 1\}$, and dilate it so it just makes contact with the l_1 unit ball, which is an m-dimensional octahedron.

Imagine we begin with a very small t, and that we increase it gradually. Considered geometrically, this corresponds to expanding the sphere $\{\mathbf{x} \,|\, \|\mathbf{x}\|_2 = t\}$, as well as its image under A, which is an m-dimensional ellipsoid, until this image, which lies fully within the m-dimensional octahedron $\{\mathbf{y} : \|\mathbf{y}\|_1 = 1\}$ with vertices at $y_i = \pm 1,\ i = 1, \ldots, m$, eventually touches the hyper-octahedron. The point of this first contact cannot be a vertex of the octahedron, because the ellipsoid is smooth and convex, so it must lie in the interior of a facet. Moreover, points of contact will occur in pairs on opposite and symmetric sides with respect to the origin. These two observations correspond to the results shown above that maximization solutions occur in pairs and that they occur within W.

The same intuition with the shape of the balls in the domain and range spaces switched yields (2c), as a vertex of the transformed octahedron from the domain space will be the first point of contact.

4 Computational complexity of $\|\cdot\|_{21}$

The calculation of $\|A\|_{21}$ when $A : \mathbb{R}^{n\times m}$, by (2d*), requires a combinatorial search, i.e., the calculation of $\|A^T\mathbf{s}\|_2$ for each $\mathbf{s} \in \{-1, +1\}^m$. This requires $O(nm2^m)$ operations.

A simple argument shows that a maximizing $\mathbf{s}$ will be in $\{-1, +1\}^m$. Let $\mathbf{s}' = \mathbf{s} - s_i\mathbf{e}_i$ and $\mathbf{a}_i$ be the ith row of A. Expanding, we obtain:

$$\|A^T\mathbf{s}\|_2^2 = \mathbf{s}^T AA^T\mathbf{s} = (\mathbf{s}')^T AA^T\mathbf{s}' + s_i^2\|\mathbf{a}_i\|_2^2 + 2s_i[(\mathbf{s}')^T AA^T\mathbf{e}_i]$$

This expression makes it clear that, keeping all elements of $\mathbf{s}$ fixed except s_i, $\|A^T\mathbf{s}\|_2$ is maximized when s_i becomes $+1$ or -1, according to whether $(\mathbf{s}')^T AA^T\mathbf{e}_i$ is positive or negative, respectively; if it is 0 it obviously makes no difference. Repeating this argument for $i = 1, \ldots, m$ proves the claim.

5 $\|\cdot\|_{pq}$ for $p \le 1 \le q$

We can generalize (2c), the well known result for the $l_1 \to l_2$ induced matrix norm, to the $l_p \to l_q$ induced matrix norm where $p \le 1 \le q$.

Theorem 2.

$$\|A\|_{pq} = \max_{j=1,\ldots,n} \|\mathbf{a}^j\|_q \qquad (p \le 1 \le q) \tag{3}$$

Proof. We use a simple geometric argument in which the roles of the balls in Figure 1 are reversed, along with considerations of convexity. We wish to find the smallest $t > 0$ such that $\exists \mathbf{x} : \|\mathbf{x}\|_p = t$ and $\|A\mathbf{x}\|_q = 1$. Imagine once more we begin with a very small t, and that we increase it gradually.

Considered geometrically, this corresponds to expanding the closed surface $P(t) = \{\mathbf{x} : \|\mathbf{x}\|_p = t\}$, and therefore $AP(t)$, the image of $P(t)$ under A, until this image, which lies fully within the closed surface $Q = \{\mathbf{y} : \|\mathbf{y}\|_q = 1\}$, touches Q.

If Q is convex, then the first point of contact between $AP(t)$ and Q will be on a point of $AP(t)$ that is on the *convex hull* of $AP(t)$. When $p < 1$, the points of $AP(t)$ on the convex hull of $AP(t)$ are in the finite set $\{\pm t\mathbf{a}^j : j = 1, \ldots, n\}$. When $q \geq 1$, the set Q is the boundary of a convex set. Together, these imply that we can find the minimal t separately for each $\{t\mathbf{a}^j\}$, and take the smallest of them. This establishes the result for $p < 1$. A limit argument based on the continuity of the norm shows that this continues to hold when $p = 1$. □

6 Induced norms on Banach spaces

How do our results need to be modified if we consider infinite-dimensional spaces instead? Since we are considering normed vector spaces, let us focus on Banach spaces, because they have the additional desirable feature of completeness. To avoid unnecessarily complicated cases, let us narrow our scope even further and exclusively consider separable Banach spaces, namely Banach spaces with a countable basis; after all, the most common ones are of this type.

Let then $A : B_1 \to B_2$ be a bounded linear operator from a separable (possibly complex) Banach space B_1 into another B_2, both equipped with (possibly different) l_p norms. Since the elements of a separable Banach space can be represented as vectors of countably many coordinates, A can be considered to be an array with countably many rows and columns (which we will also denote by A). As $\|A\| < \infty$ by assumption, then $\|A^T\| = \|\overline{A^*}\| = \|A^*\| = \|A\| < \infty$. Let now $A_{m,n} : \mathbb{R}^{n\times m}$ be the constraint of A on the first n basis vectors of B_1 followed by the projection on the first m basis vectors of B_2. We want to investigate conditions under which there exists a sequence $A_k = A_{m(k),n(k)},\ k \in \mathbb{N}^*$ such that $\lim\|A_k - A\| = 0$.

Theorem 3. *Let $A : B_1 \to B_2$ be a bounded linear operator from a separable Banach space B_1 into another B_2, equipped with the norms $\|\cdot\|_{p_1}$ and $\|\cdot\|_{p_2}$, respectively, where $1 \leq p_1, p_2 < \infty$, let $\mathbf{a}_k,\ k \in \mathbb{N}^*$ denote the rows of A, let $A_{m,n} : \mathbb{R}^{n\times m}$ be the constraint of A on the first n basis vectors of B_1 followed by the projection on the first m basis vectors of B_2, and let p_1' be such that $\dfrac{1}{p_1'} + \dfrac{1}{p_1} = 1$. If*

$$\sum_{k=1}^{\infty} \|\mathbf{a}_k\|_{p_1'}^{p_2} < \infty \qquad \text{[Row Norm Summability (RNS)]}$$

then there exists a sequence $A_k = A_{m(k),n(k)},\ k \in \mathbb{N}^*$, *such that* $\lim \|A_k - A\| = 0$.

Proof. The first step is to observe that the vector subspace of B_1 consisting of the union of the finite-dimensional subspaces of B_1 consisting of the first n basis vectors for all $n \in \mathbb{N}^*$ is dense in B_1. Similarly, the image of this subspace under A is dense in $AB_1 \subset B_2$.

The second step is to write down explicitly what $\|(A - A_k)\mathbf{x}\|_{p_2}$ is when $\|\mathbf{x}\|_{p_1} = 1$:

$$\|(A - A_k)\mathbf{x}\|_{p_2}^{p_2} = \sum_{l=1}^{m(k)} \left| \sum_{i=n(k)+1}^{\infty} a_{li} x_i \right|^{p_2} + \sum_{l=m(k)+1}^{\infty} |\mathbf{a}_k \cdot \mathbf{x}|^{p_2} .$$

However, by Hölder's inequality,

$$|\mathbf{a}_k \cdot \mathbf{x}| \leq \|\mathbf{a}_k\|_{p_1'} \|\mathbf{x}\|_{p_1} = \|\mathbf{a}_k\|_{p_1'}$$

whence

$$\|(A - A_k)\mathbf{x}\|_{p_2}^{p_2} \leq \sum_{l=1}^{m(k)} \left| \sum_{i=n(k)+1}^{\infty} |a_{li}|^{p_1'} \right|^{\frac{p_2}{p_1'}} + \sum_{l=m(k)+1}^{\infty} \|\mathbf{a}_k\|_{p_1'}^{p_2}.$$

Let now

$$m(k) = \min \left\{ m \in \mathbb{N} : \sum_{l=m+1}^{\infty} \|\mathbf{a}_k\|_{p_1'}^{p_2} < \frac{1}{2k} \right\}$$

and, having chosen $m(k)$, let

$$n(k) = \min \left\{ n \in \mathbb{N} : \max_{l=1,\ldots,m(k)} \left| \sum_{i=n+1}^{\infty} |a_{li}|^{p_1'} \right| < \left(\frac{1}{2km(k)} \right)^{\frac{p_1'}{p_2}} \right\}.$$

It follows that

$$\|(A - A_k)\mathbf{x}\|_{p_2}^{p_2} \leq \frac{1}{k} \tag{4}$$

which implies that

$$\sup_{\{\mathbf{x} \in B_1 : \|\mathbf{x}\|_{p_1} = 1\}} \|(A - A_k)\mathbf{x}\|_{p_2}^{p_2} = \|A - A_k\|_{p_2}^{p_2} \leq \frac{1}{k} \tag{5}$$

which finally results in

$$\lim_{k \to \infty} \|A - A_k\|_{p_2} = 0. \tag{6}$$

A "3-ϵ" argument, to include vectors with infinitely many nonzero coefficients over the chosen bases, finishes the proof. □

This theorem proves that the results proved in Theorems 1 and 2 remain valid for arrays of infinite dimension satisfying the RNS condition. Specifically in relation to Theorem 1, we make the following observation: Theorems 1 and 3 combined yield that

$$\|A\|_{21} = \lim_{k\to\infty} \max_{\mathbf{s}\in\{-1,+1\}^{m(k)}} \sqrt{\mathbf{s}^T A_k A_k^T \mathbf{s}} = \max_{\mathbf{s}\in\{-1,+1\}^{\infty}} \sqrt{\mathbf{s}^T A A^T \mathbf{s}}$$

despite the fact that the maximizing vector $\mathbf{s}$ where $s_i = \pm 1,\ i \in \mathbb{N}$, is no longer a vector of finite norm. Therefore, the supremum involved in the computation of $\|\cdot\|_{21}$ in this case is a genuine supremum, not a maximum, in the sense that there exists no vector $\mathbf{x}$ of finite norm, hence within the vector space, for which $\|A\|_{21} = \mathbf{x}^T A A^T \mathbf{x}$. Instead, if A is a continuous linear operator between the spaces defined in the theorem, there exists a sequence of vectors $\mathbf{x}_n,\ n \in \mathbb{N}$, such that $\lim \|A\mathbf{x}_n\|_1 = \|A\|_{21}$ and $\|\mathbf{x}_n\|_2 = 1$.

7 Summary and conclusion

Using a combination of analytical and geometrical arguments, we have shown that:

- The induced norm $\|A\|_{21}$ can be computed as the maximal value of a quadratic form over a finite, albeit exponentially sized, set of vectors (Theorem 1).
- The induced norm $\|A\|_{pq},\ p \leq 1 \leq q$ is the maximal l_q norm of the columns of A (Theorem 2).
- Analogous results hold for bounded linear operators on separable Banach spaces that satisfy RNS (Theorem 3).

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