

# Additional sampling criterion for the linear canonical transform

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The linear canonical transform describes the effect of first-order quadratic phase optical systems on a wave field. Several recent papers have developed sampling rules for the numerical approximation of the transform. However, sampling an analog function according to existing rules will not generally permit the reconstruction of the analog linear canonical transform of that function from its samples. To achieve this, an additional sampling criterion has been developed for sampling both the input and the output wave fields.

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The linear canonical transform (LCT) [1,2] is a parameterized linear integral transform that is used to relate the input and output wave fields of first-order optical systems. Recent literature has discussed discretization of the transforms [3,4], LCT sampling theorems [3,5–9] and fast algorithms [10,11]. Existing LCT sampling theory determines sampling rates for the system output wave field sufficient for reconstruction of the analog field from its samples. However, it appears the implications of sampling both the input and the output have not been discussed. This must be addressed for numerical simulation problems and the modeling of discrete optical components. In this Letter, we determine a sufficient condition on the sampling rate at the input for reconstruction of the sampled output.

We consider some definitions and concepts used in this Letter. Given a function,  $f(x)$ , the LCT of that function,  $L_T\{f(x)\}(x')$ , may be calculated as per [12],

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where  $a$ ,  $b$ ,  $c$ , and  $d$  are real parameters of the transform.  $T$  is known as the  $ABCD$  matrix or ray transfer matrix of the system and has unit determinant. The domains of the input and output functions are  $x$  and  $x'$ , respectively. The Fourier transform (FT) is a special case of the LCT. The domain reached using the FT is denoted  $k$ . The Wigner distribution function (WDF)  $W(x, k)$ , is a Cohen-class pseudodistribution with many useful properties [13,14]. The effect of an LCT on the WDF of a function is given by

$$W(x, k) \rightarrow W(x', k'),$$

$$\begin{pmatrix} x' \\ k' \end{pmatrix} = T \begin{pmatrix} x \\ k \end{pmatrix}. \quad (1)$$

The  $(x, k)$  plane is commonly referred to as “phase space.”

A function’s WDF may be treated as being completely contained—bounded by some shape. This is not strictly accurate [12] but a valid approximation in general. The area of this shape is the space-bandwidth product (SBP) and is a measure of the information carrying capacity of the function. One such shape is a rectangle centered on the origin. This rectangle is of width  $2B$  (the function’s bandwidth) in the  $k$  direction and length  $2W$  (the function’s support) in the  $x$  direction. We refer to this rectangle as the phase space “footprint” of the function. Such diagrams are known as “phase space diagrams” (PSDs) or “Wigner charts.” Analyses based on them neglect the cross terms inherent in distributions such as the WDF.

We will use PSDs to develop a sampling rule for sampling both the input and the output of an LCT. For illustration and comparison, we first discuss the FT.

We sample a function,  $f(x)$ . This is typically described as multiplication with a “comb” function [15,16],

$$f_d(x) = \sum_{n=-\infty}^{\infty} f(x) \delta(x - nT_x). \quad (2)$$

For the FT, the sampling rate,  $T_x$ , is determined by the Shannon sampling theorem [17], which requires  $T_x \leq 1/2B$ , the reciprocal of which is the Nyquist sampling rate. The effect in phase space of this sampling is to produce periodic replicas in the direction orthogonal to the sampling domain, as multiplication by the comb function is equivalent to convolution with another comb function in the Fourier domain; see Fig. 1. Note that all figures in this Letter are illustrative and not quantitative. The spacing of the replicas depends on  $T_x$ .

The Shannon requirement can be stated as requiring that the replicas on either side do not overlap the zeroth order, i.e., the copy centered on the origin. This permits the recovery of the analog function by filtering.

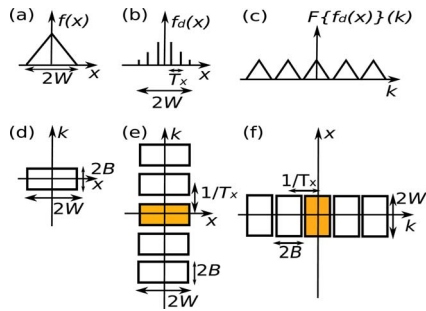


Fig. 1. (Color online) (a)  $f(x)$ . (b)  $f_d(x)$ . (c) FT of  $f_d(x)$ . (d) PSD of  $f(x)$ . (e) PSD of  $f_d(x)$ . (f) PSD of FT of  $f_d(x)$ . Zeroth-order replica shaded.

$f(x)$  is approximated by a finite number of its samples, i.e., the sum in Eq. (2) is truncated to a length  $2W$ , the support of the function.  $2W$  and  $T_x$  define the phase space footprint of  $f(x)$ . Assuming  $N = 2W/T_x$  samples, the discrete-space FT of the sampled function is

$$F\{f_d(x)\}(k) = \sum_{n=-N/2}^{N/2} f_d(nT_x) \exp(-jnT_x k). \quad (3)$$

This analog function consists of the superposition of regularly shifted copies of the FT of  $f(x)$ . Note that the effect of the FT on the PSD is to rotate it by  $90^\circ$ ; see Fig. 1. The discrete-space FT is sampled,

$$F_d\{f_d(x)\}(mT_k) = \sum_{n=-N/2}^{N/2} F\{f_d(x)\}(k) \delta(k - nT_k). \quad (4)$$

The required output sampling rate is determined by the support of the input function as rotating phase space  $90^\circ$  results in the support of the input becoming the bandwidth of the output.  $T_k = 1/NT_x = 1/2W$ , where  $N$  is the number of samples used. As a result of this relationship, Eq. (4) is usually written with  $T_k$  in terms of  $N$  and  $T_x$ . Sampling the output function, Eq. (4), again produces replicas in phase space in the direction orthogonal to the sampling domain, Fig. 2. The zeroth-order replica (shaded) is recoverable using simple filtering operations in both  $k$  and  $x$ .

We now consider the sampling process for the general case of the LCT. We start with Eq. (2). The problem of selecting  $T_x$  will be returned to in light of later steps in this process. The sampled function is truncated as before, and its LCT is (assuming  $b \neq 0$ )

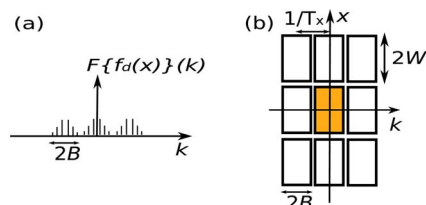


Fig. 2. (Color online) (a) Sampled FT of  $f_d(x)$ . (b) PSD of this.

$$\begin{aligned} L_T\{f_d(x)\}(x') &= \frac{e^{j\pi/4}}{\sqrt{2\pi b}} \sum_{n=-N/2}^{N/2} f_d(nT_x) e^{(j/2b)[a(nT_x)^2 - 2nT_x x' + dx'^2]}. \end{aligned} \quad (5)$$

This is periodic in phase space, as shown in Fig. 3. We can recover the original function using the methodology of [3,5,6,9]. However, choosing an appropriate sampling rate for the output poses a problem. In phase space, this sampling of the output produces shifted replicas of Fig. 3(d) in the  $k'$  domain, as shown in Fig. 4. Figure 4(c) illustrates why it may be necessary to sample the input at a rate greater than the Nyquist rate. If we chose our rates incorrectly, replicas may overlap the zeroth order, making reconstruction of the analog function less accurate.

We wish to determine a sampling rate so that this overlap does not occur, as shown in Fig. 4(d). One solution is to require that the vertical dashed lines in Fig. 4(d) pass either side of the zeroth-order replica and not cut any other replica created by the sampling in the input domain, as follows. Consider a point in the phase space footprint of a sampled function,  $p_1(x'_1, k'_1)$ . The equivalent point in one of the two nearest replicas of the footprint is given by  $p_2[x'_1, k'_1 + (1/T_x)]$ . These points are transformed by Eq. (1). This results in the points  $p'_1(ax'_1 + bk'_1, cx'_1 + dk'_1)$  and  $p'_2[ax'_1 + bk'_1 + (b/T_x), cx'_1 + dk'_1 + (d/T_x)]$ . These points are separated by the vector  $(b/T_x, d/T_x)$ . We require that the horizontal separation of these transformed points be greater than the extent of the transformed (unsampled) function,  $L_T\{f(x)\}(x')$ . Similar results can be found for all other replicas, with this one defining the lower bound on the sampling rate,

$$T_x \leq \frac{|b|}{2\hat{W}}, \quad (6)$$

where  $2\hat{W}$  is the extent of  $L_T\{f(x)\}(x')$  in the domain  $x'$ . This bound results in the situation shown in Fig. 4(d). It is important to note that this bound must be used *in addition to* the Nyquist criterion. We note that Eq. (6) does not constrain  $T_x$  beyond the Nyquist period in cases where one or more of the transform

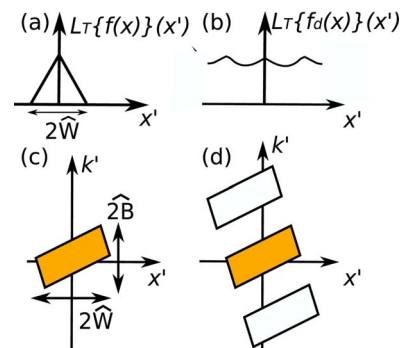


Fig. 3. (Color online) (a) LCT of  $f(x)$ . (b) LCT of  $f_d(x)$ . Significant aliasing is present. (c) PSD of the LCT of  $f(x)$ . (d) PSD of the LCT of  $f_d(x)$ . The zeroth order is shaded.

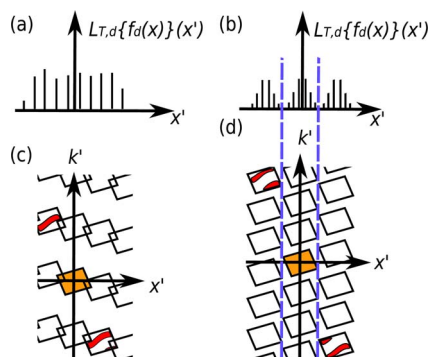


Fig. 4. (Color online) (a) The sampled LCT of the undersampled input. (b) The sampled LCT of the correctly sampled input. (c) The PSD of the sampled LCT of the undersampled input. The zeroth-order term is shaded. The terms created by sampling the input are patterned. (d) The PSD of the sampled LCT of the correctly sampled input.

parameters is near zero. However, the analysis above based on phase space diagrams continues to be useful. For the  $b=0$  case, the output domain is discrete if the input is, so the case is covered by existing sampling theorems. A method of calculating  $\hat{W}$  is implied in [7,8] and explicitly determined in [4]. We emphasize that Eq. (6) is not the only possible criterion here. Any pair of input and output sampling rates that keep the replicas from overlapping are sufficient.

If the input is chosen as above, the output sampling rate may be chosen using previous sampling theorems. For example, as per [8], we may require the output sampling period to be not less than  $2\hat{B}$ , which is sufficient to separate the central copy from its own replicas using a low-pass filter.  $2\hat{B}$  may be found similarly to  $2\hat{W}$ .

In conclusion, to numerically approximate the LCT of a function, we must sample it in both the input and the output domain. Sampling in the output domain places a further constraint on the input sampling rate. We have established an upper bound on the necessary rate, and an appropriate reconstruction

method. The analysis presented in this Letter has significant consequences for numerical LCT algorithms, and it may also be possible to use this work to generalize the Gerchberg–Saxton iterative phase retrieval algorithm. There may also be consequences in digital holography and other techniques that use CCDs or digital cameras to sample and capture monochromatic fields.

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