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Universality and scaling in chaotic attractor-to-chaotic attractor transitions

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Abstract

In this paper we discuss chaotic attractor-to-chaotic attractor transitions in two-dimensional multiparameter maps as an external parameter is varied. We show that the transitions are sharply defined and may be classed as second-order phase transitions. We obtain scaling laws, about the critical point A_c, for the average positive Lyapunov exponent, $(\lambda^+ - \lambda_c^+) \sim |A - A_c|^\beta$, where λ_c^+ is the value of the positive Lyapunov exponent at crisis, and the average crisis induced mean lifetime $\tau \sim |A - A_c|^{-\gamma}$, where A is the parameter that is varied. Here average means averaged over many initial conditions. Furthermore we find that there is an algebraic relationship between the critical exponents and the correlation dimension D_c at the critical point A_c, namely, $\beta + \gamma + D_c =$ constant. We find this constant to be approximately 2.31. We postulate that this is a universal relationship for second-order phase transitions in two-dimensional multiparameter non-hyperbolic maps. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

A central problem in non-linear dynamics is that of discovering how the properties of orbits change and evolve as a parameter of a dynamical system is changed [1]. The great advances in non-linear physics in the last thirty years have followed from such an approach. For the logistic map as a single parameter in the map is varied there is a period doubling cascade terminating in an accumulation in an infinite number of period doublings, followed by a parameter range in which chaos and periodic windows are intermixed [1–3]. Other cases are frequency locking and Arnold tongues [4–6] and the transition from quasiperiodicity to chaos [1,2,7]. The simple chaos to periodic Manneville Pomeau type intermittency of the logistic map has given us a fundamental theory of certain types of $1/f$ noise [8,9]. These scenarios have revolutionized our understanding of non-linear physics. They all involve a sequential transition from periodic to chaotic behaviour or vice versa [1,2].

Another very important and physically relevant transition is one where the system is in a chaotic state and as a parameter is changed we go from one type of chaos to another [1,10]. Such transitions are a very common feature of many physical (non-hyperbolic) systems such as the forced damped pendulum, the Duffing equation and the Henon map. However very little is understood about such transitions. The term crisis induced intermittency was introduced in the pioneering work of Grebogi et al. [10,11] to describe a certain type of $\text{chaos}_1 \Longleftrightarrow \text{chaos}_2$ transition. They found that the characteristic transient lifetime, τ, scales as $\tau \sim |A - A_c|^{-\gamma}$. They determined two types of crises for these two-dimensional maps and found a relationship between γ and the eigenvalues of the orbit responsible for the crisis.

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1.1. Heteroclinic tangency crisis

In this case the stable manifold of an unstable periodic orbit at the edge of the basin of attraction of the attractor becomes tangent to the unstable manifold of an unstable periodic orbit on the attractor. The critical exponent in this case is given by

$$\gamma = \frac{1}{2} + \frac{|\ln \epsilon^{+}|}{|\ln \epsilon_{-}|},$$

where ϵ^{+} and ϵ_{-} are the expanding and contracting eigenvalues of the Jacobian matrix of the map evaluated for the unstable periodic orbit on the attractor.

1.2. Homoclinic tangency crisis

In this case the stable and unstable manifolds of an unstable periodic orbit on the boundary of the basin of attraction are tangent. The critical exponent is given by,

$$\gamma = \frac{\ln |\epsilon^{+}|}{\ln |\epsilon_{-}\epsilon^{+}|^{2}},$$

where ϵ^{+} and ϵ_{-} are now the expanding and contracting eigenvalues of the Jacobian matrix of the map evaluated for the unstable periodic orbit on the boundary of the attractor [1,10,11].

In this paper we present the results of a detailed investigation of the phenomenon of crisis induced intermittency [10] in discrete two-dimensional multiparameter mappings from a statistical-thermodynamic approach. All the mappings that we study in this paper are non-hyperbolic. They exhibit, for certain parameter ranges, the classic intermittent behaviour investigated by Grebogi et al. [10,11]. In all cases the intermittency is a direct consequence of a crisis as one of the system parameters, say A, passes through some critical value A_c. In all the mappings studied there is a qualitative similarity between the behaviours both before and after the crisis has occurred. For all the systems that we investigate there is a sudden widening in phase space volume of the attractor for the dynamical system as the parameter passes through the critical point, A_c. In these mappings however, the attractor which exists before the crisis, consists, in each case, of a number of separate pieces, the precise number of these pieces being dependent upon the dynamical system in question. An orbit on these fragmented attractors moves periodically among the individual pieces and this behaviour is universal for all the mappings studied in this paper.

Once the varied parameter A has passed through the critical value A_c, the dynamics of the system is intermittent. An orbit on the attractor spends a length of time in the old region to which the dynamics was confined before the crisis occurred. At the end of one of these stretches of time, the orbit suddenly bursts out of the old region and spends some time in the larger phase space region in which the dynamics is now confined, post crisis. We obtain the average Lyapunov exponents at given values of A of the map on both sides of the 'phase' transition point A_c. Here average means averaged over many initial conditions. Unlike the work of Grebogi, we are dealing with global averages over many initial conditions and not with specific individual orbits responsible for the crisis. However we find that these global averages also scale across the transition point. In particular, we find that the positive Lyapunov exponent obeys a similar scaling law to that obeyed by τ and we evaluate the corresponding critical exponent.

Firstly, we investigate the validity of the scaling relation found by Grebogi et al. [10] for the maps studied in this paper, namely, $\tau \sim |A - A_c|^{-\gamma}$, via the calculation, in each case, of the scaling parameter γ. Here, τ represents the mean time spent by the attractor in the region to which the attractor was confined before the crisis occurred. For a given parameter value A the value of τ is calculated in the parameter range where the dynamics of the system is intermittent.

Secondly we examine the behaviour of the average positive Lyapunov exponent as a system parameter, A, is varied through the critical value A_c. We find that it varies as $(\lambda^{+} - \lambda_c^{+}) \sim |A - A_c|^{\beta}$, where λ^{+} is the positive Lyapunov exponent averaged over many initial conditions, and λ_c^{+} is the value of the positive Lyapunov exponent at A_c. We test the validity of this result by calculating the scaling parameter β for all cases discussed in the paper. Finally, we discover that there is a fundamental relationship between the critical exponents and the correlation dimension at the critical phase transition point, which is the analogue of one of the Rushbrooke Fisher Widom Josephson relations for second-order phase transitions in the theory of critical phenomena [12]. We test the relationship numerically for many different two-dimensional multiparameter maps, including the generalised Ushiki and Mira maps.

2. The Ushiki map

The phase transitions that we discuss can be illustrated by the Ushiki and Ikeda maps, which we will discuss in detail. The other maps that we refer to in the paper are summarized in Appendix A. Appendix A contains details of the Mira maps and generalized Ushiki map and details of the specific phase transitions referred to in the paper. The Ushiki map [13] is a two-dimensional map given as follows:

$$X_{i+1} = (A - X_i - B_1 Y_i)X_i,$$
$$Y_{i+1} = (A - Y_i - B_2 X_i)Y_i.$$

The parameters A, B_1 and B_2 are chosen such that $A > 1$ and $0 < B_1, B_2 < 1$. In the analysis which follows, B_1 and B_2 were assigned the values 0.1 and 0.15, respectively, and A is varied.

The system was then investigated in the parameter range $3.7457092386 < A < 3.7457092387$, where the critical value of the parameter A occurs in this interval. For $A > A_c$ the attractor of the system has an eight piece structure as in Fig. 1. Each of these individual attractors has the same qualitative features and also a similar multifractal structure. One of these attractors is illustrated in Fig. 2.

As the value of the parameter A passes through A_c a crisis occurs and the attractor suddenly expands in phase space. The new attractor is shown in Fig. 3.

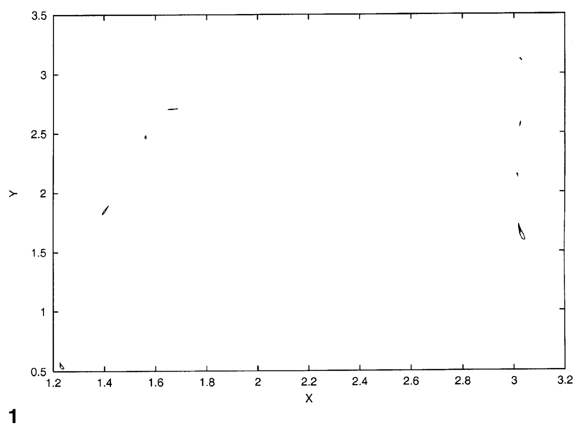
A typical orbit on this attractor behaves intermittently, switching from the phase space regions which previously contained the eight piece attractor described above, and the new enlarged phase space region made available by the crisis.

2.1. Intermittency scaling exponent

For $A < A_c$ and A close to A_c the dynamical behaviour of the system is intermittent. As shown by Grebogi et al. [10] the dependence of τ upon A is given by $\tau \sim |A - A_c|^{-\gamma}$, where τ is the mean time spent by an orbit, for a given value of the parameter A, in the region of phase space that the attractor was confined to prior to the occurrence of the crisis. An estimate of γ for the Ushiki map was obtained by Jenkins [14]. He found, and we also find, that $\gamma \simeq 0.78 \pm 0.02$.

2.2. Positive Lyapunov exponent and critical scaling exponent

By examining the spectrum of Lyapunov exponents one can quickly determine the range of A values for which the behaviour of the dynamical system is chaotic. If at least one of the Lyapunov exponents is positive, then the system displays sensitivity to initial conditions – a hallmark of chaotic dynamics. For a range of A values close to A_c a plot of the positive Lyapunov exponent is shown in Fig. 4. It is obvious that λ^+ can serve as an order parameter for the transition. Examination of this plot suggests that the positive Lyapunov exponent obeys a similar type of scaling law to τ as described in the previous section, i.e., $(\lambda^+ - \lambda_c^+) \sim |A - A_c|^{\beta}$. The value of λ_c^+ is found to be $\lambda_c^+ \simeq 0.0197$. We tested



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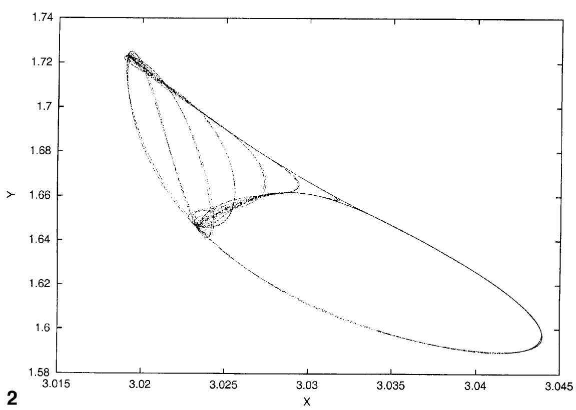

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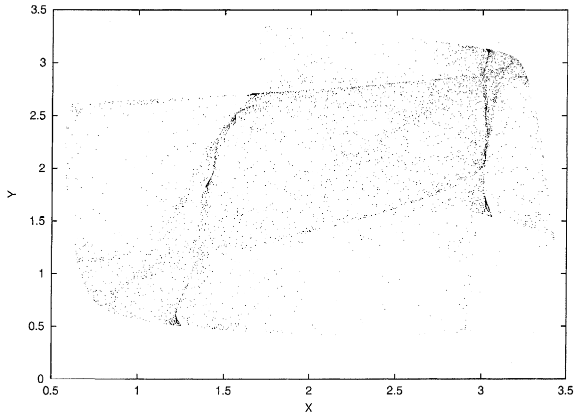
Fig. 1. Eight piece Ushiki attractor before crisis for $A = 3.7457092387$.

Fig. 2. One of the eight pieces of the Ushiki attractor before crisis for $A = 3.7457092387$.



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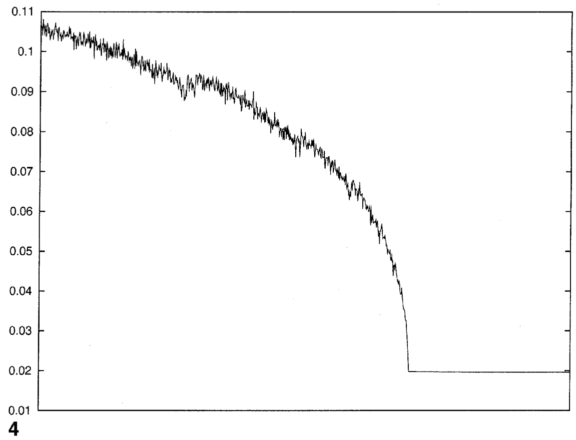

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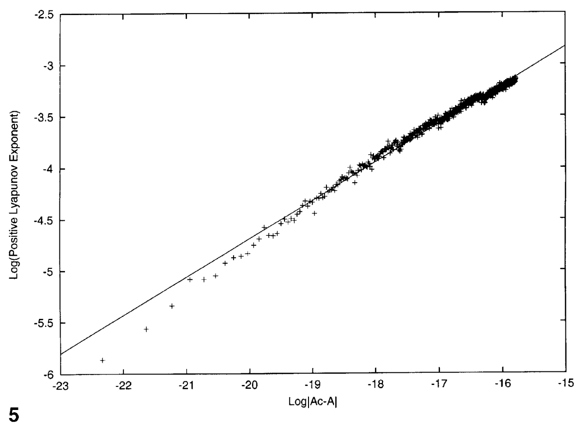
Fig. 3. Enlarged Ushiki attractor after the crisis for $A = 3.7457092386$.

Fig. 4. Positive Lyapunov exponent plotted as a function of the parameter A as A is varied through the critical value $A_c : 3.7457091 \leqslant A \leqslant 3.7457093$.



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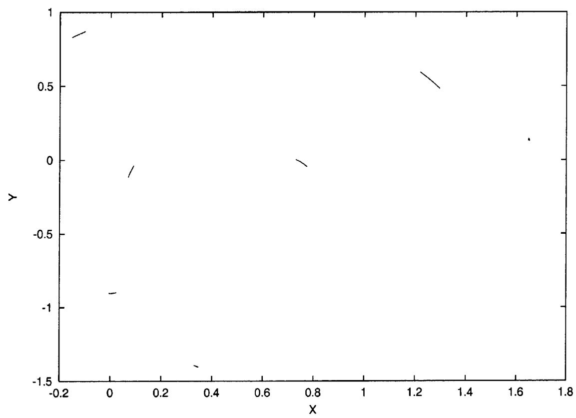

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Fig. 5. Plot of $\log(\lambda^+ - \lambda_c^+)$ against $\log(A_c - A)$ for the data in Fig. 4, where λ^+ is the positive Lyapunov exponent.

Fig. 6. Seven piece Ikeda attractor before the crisis for $\eta = 8.984902$.

this hypothesis and the results of a log–log plot of $\log(\lambda^+ - \lambda_c^+)$ against $\log|A - A_c|$ together with the least squares fit line are shown in Fig. 5. The fitted value of β is $\beta \simeq 0.37$.

3. The Ikeda map

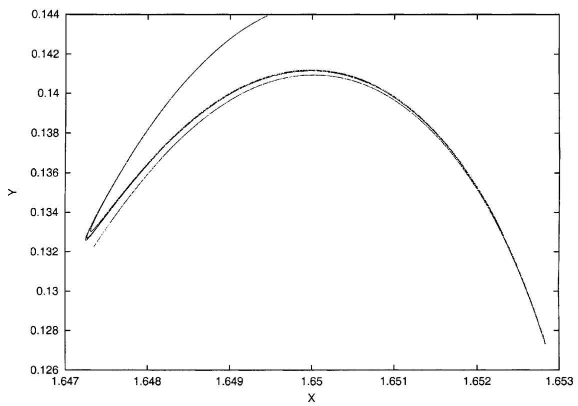
Another map which displays a second-order phase transition is the Ikeda map. The Ikeda map is given by

$$Z_{n+1} = A + BZ_n \exp(\imath\kappa - \imath\eta/(1 + |Z_n|^2)),$$

where $Z = x + \imath y$ is a complex number.

This map models the behaviour of an optical bistable ring cavity laser system [15]. The mapping can be studied as a real two-dimensional map in the variables x and y by making the substitution $Z = x + \imath y$. The resulting map is given by

$$x_{n+1} = A + Bx_n \cos\phi_n - By_n \sin\phi_n,$$
$$y_{n+1} = Bx_n \sin\phi_n + By_n \cos\phi_n,$$



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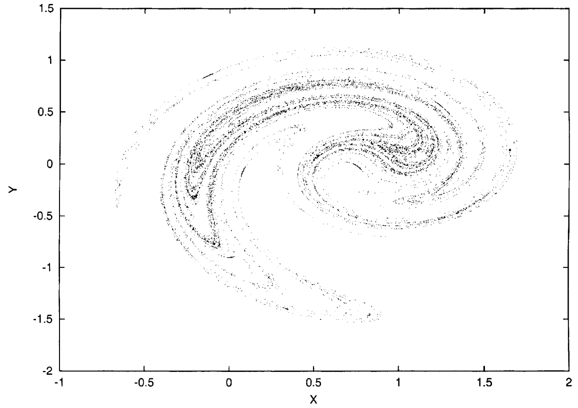

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Fig. 7. One of the seven pieces of the Ikeda attractor before the crisis for $\eta = 8.984902$.
Fig. 8. Enlarged Ikeda attractor after the crisis for $\eta = 8.984901$.

where

$$\phi_n = \kappa - \eta/(1 + x_n^2 + y_n^2).$$

We have investigated this map for the following set of parameters: $A = 0.85$, $B = 0.9$, $\kappa = 0.4$ and we varied the parameter η about the critical value $\eta_c \simeq 8.9849019$.

For $\eta > \eta_c$ the attractor consists of seven separate attractors as shown in Fig. 6. An enlarged phase plot of one of these seven qualitatively similar attractors is shown in Fig. 7. They all posses a very similar fractal structure.

Again as with the Ushiki map, as η passes through the critical parameter value, η_c, a crisis occurs which induces intermittency in the dynamical behaviour of the system. For this value of η, the attractor has become enlarged as shown in Fig. 8.

3.1. Intermittency scaling exponent

For $\eta < \eta_c$ the dynamics on the attractor in Fig. 8 is intermittent.

In the spirit of the conjecture of Grebogi et al. [10], we tested the scaling hypothesis for τ the mean time spent by an orbit in the region of the seven piece attractor displayed in Fig. 7. The scaling hypothesis is that $\tau \sim |\eta - \eta_c|^{-\gamma}$. The results of a plot of $\log \tau$ against $\log |\eta - \eta_c|$ are shown in Fig. 9, together with the least squares fitted straight line. The fit gives an estimate for γ to be $\gamma \simeq 0.759 \pm 0.06$.

3.2. Positive Lyapunov exponent and critical scaling exponent

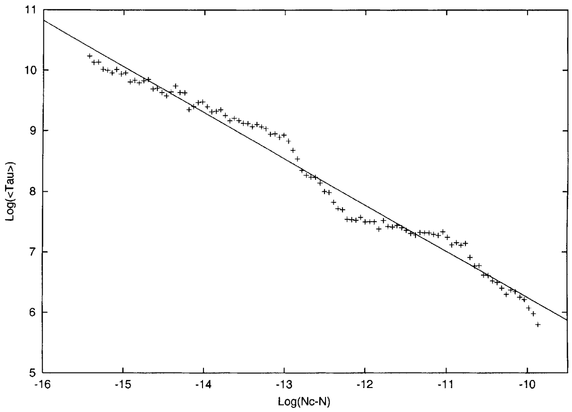
Again by examining the Lyapunov spectrum one can ascertain whether the dynamical behaviour is chaotic or not. In Fig. 16 a plot of the positive and negative Lyapunov exponents of the Ikeda map is shown for a range of η values about the critical value η_c. In Fig. 10 a plot of the Kaplan–Yorke dimension ($D_1 = 1 + (\lambda^+/|\lambda_-|)$) [16] is shown for a range of η values near the critical value η_c. From the plots in both of these figures one can see clearly the classic indicator of a phase transition – a discontinuity in an 'order parameter'.

We find the following scaling relation for the positive Lyapunov exponent. As with the Ushiki map it had the form $(\lambda^+ - \lambda_c^+) \sim |\eta - \eta_c|^{\beta}$.

In Fig. 11 a plot of $\log(\lambda^+ - \lambda_c^+)$ against $\log |\eta - \eta_c|$ is shown together with the fitted least squares line ($\lambda_c^+ \simeq 0.458$). The estimate of the scaling exponent β from this procedure is $\beta \simeq 0.035$.

4. Correlation dimension and the relationship between the critical exponents

We have calculated the correlation dimension for the above maps at the transition points [17,18]. The correlation sum $C(r)$ counts the points that fall within a radius r of a given point,



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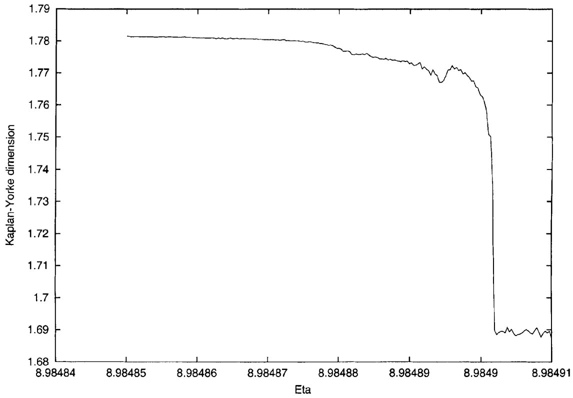

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Fig. 9. Shown is a plot of $\log\tau$ vs $\log(\eta_c - \eta)$ for the Ikeda map together with the least-squares fitted straight line; (τ is the mean time spent by an orbit in the region of the seven piece attractor for $\eta < \eta_c$).

Fig. 10. The Kaplan–Yorke dimension as a function of η as η is varied the critical value η_c.

$$C(r) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} \Theta(r - |x_i - x_j|),$$

where $\Theta(x)$ is the Heaviside step function,

$$\Theta(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geqslant 0. \end{cases}$$

The correlation dimension is defined by

$$C(r) = \lim_{r\to 0} k r^{D_c},$$

where k is an arbitrary constant and thus D_c acts like a 'critical exponent' for the separation of points on the attractor. The correlation dimension is calculated from

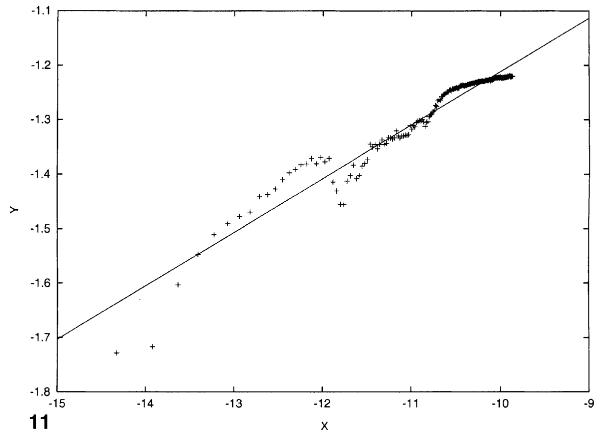
$$D_c = \lim_{r\to 0} \frac{\log C(r)}{\log r}.$$

A summary of the values of the critical exponents γ, β and D_c is listed in Table 1 for the above maps and for the generalised Ushiki, Mira1 and Mira2 maps, which exhibit the same type of phase transition. (These multiparameter two-dimensional maps and their phase transition points are summarised in Appendix A.)

We find that these critical exponents are related by $\gamma + \beta + D_c = \text{constant}$. We find this constant to be approximately 2.31 for the mappings studied in this paper with one exception. The exception is the Mira1 map which, unlike the other maps, is exceptional in that is has a one-dimensional structure in the phase space. We postulate that this is a universal relationship for multiparameter two-dimensional non-hyperbolic maps of the type discussed in this paper.

Table 1
A complete listing of the scaling exponents γ and β and the correlation dimensions for each of the maps, together with their sums

Map	γ	β	D_c	Sum
Ushiki	0.78 ± 0.02	0.37 ± 0.01	1.14 ± 0.01	2.29
Ikeda	0.76 ± 0.06	0.035 ± 0.01	1.59 ± 0.01	2.38
Mira1	0.61 ± 0.04	0.38 ± 0.002	0.94 ± 0.01	1.92
Mira2	0.54 ± 0.01	0.70 ± 0.001	1.03 ± 0.01	2.27
Generalized Ushiki	0.77 ± 0.02	0.39 ± 0.01	1.17 ± 0.01	2.33



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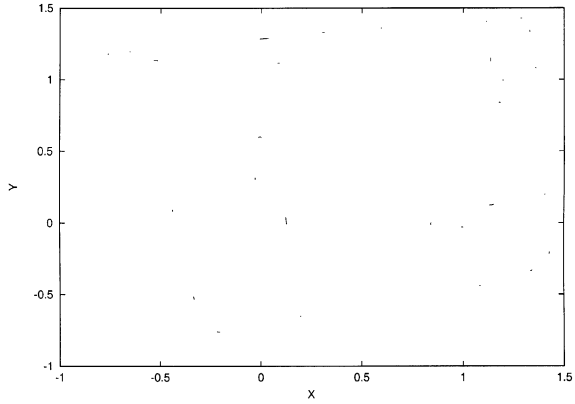

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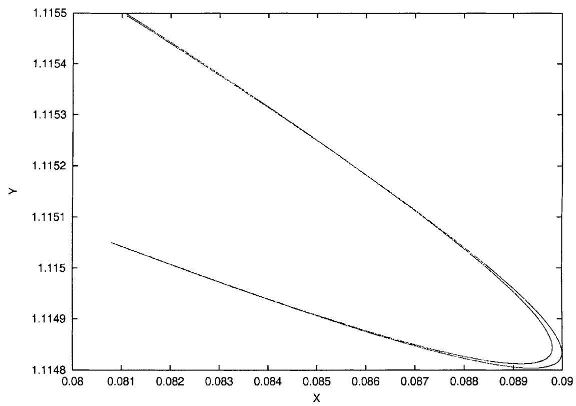
Fig. 11. Plot of $\log(\lambda^+ - \lambda_c^+)$ against $\log(\eta_c - \eta)$ for the Ikeda map, where λ^+ is the positive Lyapunov exponent.
Fig. 12. Twenty-eight piece Mira1attractor before the crisis for $B = 1.3000324$.



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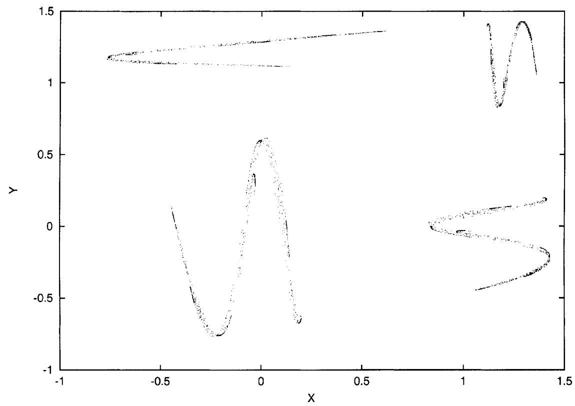

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Fig. 13. One of the 28 pieces of the Mira1 attractor before the crisis for $B = 1.3000324$.
Fig. 14. Enlarged four piece Mira1 attractor after the crisis for $B = 1.3000324$.

5. Conclusions

In conclusion, we have discussed chaotic attractor-to-chaotic attractor transitions in two-dimensional multiparameter maps as an external parameter is varied. We showed that the transitions are sharply defined and may be classed as second-order phase transitions. For the maps discussed we observed the following scaling law, about the critical point A_c, for the average positive Lyapunov exponent, $(\lambda^+ - \lambda_c^+) \sim |A - A_c|^\beta$ and, following Grebogi et al. [10,11] we showed that the average crisis induced mean lifetime, τ, also scaled as $\tau \sim |A - A_c|^{-\gamma}$. Furthermore we found that there is an algebraic relationship between the critical exponents and the correlation dimension D_c across the critical point A_c, namely, $\beta + \gamma + D_c =$ constant. We find this constant to be approximately 2.31. We postulate that this is a universal relationship for two dimensional multiparameter non-hyperbolic maps.

Acknowledgements

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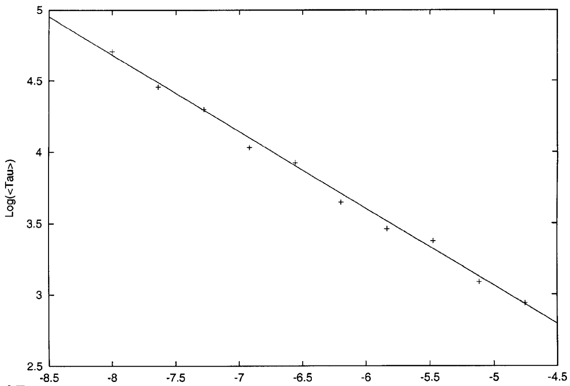

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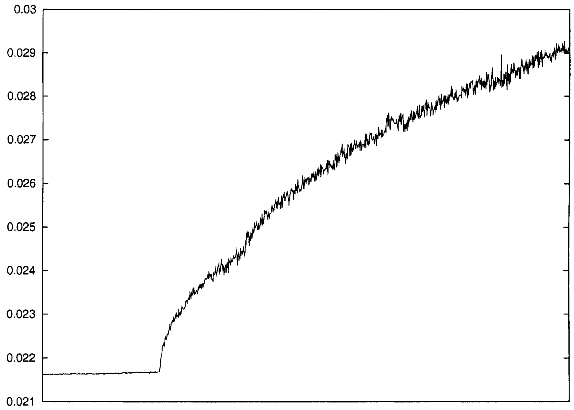

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Fig. 15. Shown is a plot of $\log \tau$ vs $\log(B - B_c)$ for the Mira1 map together with the least-squares fitted straight line; (τ is the mean time spent by an orbit in the region of the twenty eight piece attractor, for $B > B_c$).

Fig. 16. Positive Lyapunov exponent plotted as a function of the parameter B, as B is varied through the critical value B_c.

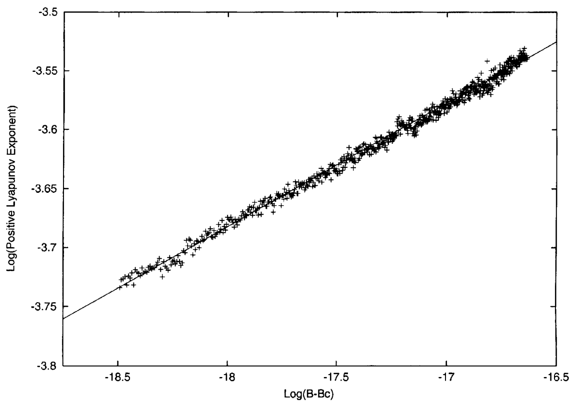

Fig. 17. Plot of $\log(\lambda^+ - \lambda_c^+)$ against $\log(B - B_c)$ for the Mira1 map, where λ^+ is the positive Lyapunov exponent.

Appendix A

Here is a summary of the other two-dimensional maps studied in this paper. They are the Mira1 and Mira2 maps and the generalised Ushiki map.

The Mira1 map [19] is given by the two-dimensional mapping

$$x_{n+1} = y_n,$$
$$y_{n+1} = Ay_n - x_n^2 + B.$$

The parameter A is fixed with the value, 0.1 and the parameter B is varied through the critical value, $B_c \simeq 1.30003242$. For $B < B_c$ the attractor consists of a fragmented twenty eight piece attractor as seen in Fig. 12. Each of these pieces has a similar fractal structure and an enlargement of one of the 28 pieces is shown in Fig. 13. An orbit on this attractor cycles in a particular sequence through each of the 28 pieces in turn so the attractor as a whole has period 28.

As B passes through B_c there is a sudden widening of the attractor in phase space accompanied by intermittent dynamics. The attractor for this B-value is shown in Fig. 14. As can be seen from the phase plot, it consists of four separate pieces each of which contains seven of the 28 separate attractors which exist for $B > B_c$.

This attractor is a period-4 attractor as an orbit on the attractor cycles in sequence through each of the four separate pieces. The period-1 fixed points of this map are

$$\left(\frac{(A-1)+\sqrt{(1-A)^2+4B}}{2}, \frac{(A-1)+\sqrt{(1-A)^2+4B}}{2}\right),$$
$$\left(\frac{(A-1)-\sqrt{(1-A)^2+4B}}{2}, \frac{(A-1)-\sqrt{(1-A)^2+4B}}{2}\right).$$

A.1. Intermittency scaling exponent

As with the previous mappings, we decided to test the intermittency scaling hypothesis for this map. Recall that the scaling relation claims that $\tau \sim |B - B_c|^{-\gamma}$. By a direct inspection of the time-series for a range of B-values above the crisis value, $B_c = 1.30003242$, a set of values for $\langle\tau\rangle$ could be obtained and consequently a plot of $\log \tau$ versus $\log |B - B_c|$ is found and shown in Fig. 15. The least-squares value for the slope of the fitted line gives the estimate for the scaling exponent γ to be $\gamma \simeq 0.54 \pm 0.01$.

It appears that this scaling relation describes very accurately the dependence of τ upon the value of the varied parameter B.

A.2. Positive Lyapunov exponent and critical exponent for the Mira1 map

In Fig. 16 a plot of the positive Lyapunov exponent, λ^+ as a function of the parameter B as it varies through the critical value B_c is shown.

Again from this plot the trademark signature of a phase transition via the non-analytic variation of an 'order parameter' is quite evident, i.e., as B increases through B_c there is a non-analytic change in λ^+.

For $B > B_c$ it appears that $(\lambda^+ - \lambda_c^+)$ scales according to the relation $(\lambda^+ - \lambda_c^+) \sim |B - B_c|^{\beta}$. Again we tested this hypothesis by plotting $\log(\lambda^+ - \lambda_c^+)$ against $\log |B - B_c|$ and the results are shown in Fig. 17 together with the fitted straight line ($\lambda_c^+ \simeq 0.023$). From the least squares fit the estimated value for β is $\beta \simeq 0.70 \pm 0.01$. Hence the positive Lyapunov exponent obeys the scaling relation given above for this map.

A.3. The Mira2 map

The Mira2 map is given by the equations

$$x_{n+1} = Ax_n + y_n,$$
$$y_{n+1} = x_n^2 + B.$$

The parameter A is fixed, $A = -1.5$ and the parameter B is varied through the phase transition point $B_c = -2.05012268$. For $B < B_c$ there is a three piece chaotic attractor. For $B > B_c$ there is a 108 piece chaotic attractor. The β, γ and D_c values for this transition are included in Table 1. Note that $\lambda_c^+ \simeq 0.013$.

A.4. Generalised Ushiki map

The generalised Ushiki map is given by

$$X_{i+1} = (A_1 - X_i - B_1 Y_i)X_i,$$
$$Y_{i+1} = (A_2 - Y_i - B_2 X_i)Y_i.$$

The following parameters are fixed: $A_2 = 3.1$, $B_1 = 0.06$ and $B_2 = 0.4$. The parameter A_1 is varied through the transition point $A_{1C} = 3.7711124$. For $A_1 < A_{1C}$ there exists a five piece chaotic attractor. When $A_1 > A_{1C}$ there is a one piece spread out chaotic attractor. The β, γ and D_c values for this transition are included in Table 1. Note that $\lambda_c^+ \simeq 0.126$.

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