CARTAN INVARIANTS AND CENTRAL IDEALS OF GROUP ALGEBRAS

Thomas Breuer
Lehrstuhl D für Mathematik
RWTH Aachen
Germany
sam@math.rwth-aachen.de

and

László Héthelyi
Department of Algebra
Budapest University of Technology and Economics
H-1521 Budapest
Műegyetem Rkp. 3-9
Hungary
hethelyi@math.bme.hu

and

Erzsébet Horváth
Department of Algebra
Budapest University of Technology and Economics
H-1521 Budapest
Műegyetem Rkp. 3-9
Hungary
he@math.bme.hu

and

Burkhard Külshammer
Mathematisches Institut
Friedrich-Schiller-Universität
07740 Jena
Germany
kuelshammer@uni-jena.de

and

John Murray
Mathematics Department
National University of Ireland
Maynooth, C. Kildare
Ireland
John.Murray@maths.may.ie

Abstract. In this paper we investigate general properties of Cartan invariants of a finite group G in characteristic 2. One of our results shows that the Cartan matrix of G in characteristic 2 contains an odd diagonal entry if and only if G contains a real element of 2-defect zero. We also apply these results to 2-blocks of symmetric groups and to blocks with normal or abelian defect groups. The second part of the paper deals with annihilators of certain ideals in centers of group algebras and blocks.

Subject Classification: 20C05, 20C20.

1. Introduction

In [9], we investigated certain ideals in the center of a symmetric algebra and obtained some consequences for the Cartan matrices of these algebras. In this paper, we specialize our results to group algebras.

In the following, let F be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. Suppose first that $p = 2$. One of our main results (Theorem 3.3) asserts that the Cartan matrix of the group algebra FG contains an odd diagonal entry if and only if G contains a real element of 2-defect zero. In fact, we can be more precise: Suppose that P is an indecomposable projective FG-module, and denote the sum in FG of all real elements of 2-defect zero in G by R_G^+. Then $\dim\ \mathrm{End}_{FG}(P)$ is odd if and only if R_G^+ does not annihilate P. Moreover, the number of blocks of FG with an odd diagonal Cartan invariant equals the dimension of the ideal $R_G^+ \cdot \mathbf{Z}FG$ of the center $\mathbf{Z}FG$ of FG.

In Section 2, we apply these results to specific groups and blocks. We prove that the symmetric group S_n of degree n has an odd diagonal Cartan invariant in characteristic 2 for every $n \geq 3$, and we show that a 2-block B of S_n has an odd diagonal Cartan invariant if and only if the weight of B is even.

We demonstrate that diagonal Cartan invariants for a 2-block with normal defect group are always even, except when the 2-block has defect zero. This fact is related to Broué's Abelian Defect Group Conjecture. We prove also that, in a finite group with elementary abelian Sylow 2-subgroups, all diagonal Cartan invariants are even, except for those coming from 2-blocks of defect zero. It should be noted that our proof does not make use of the classification of these groups.

In the second part of the paper, we return to the assumption that F is an algebraically closed field of arbitrary characteristic $p > 0$, and we investigate certain ideals in $\mathbf{Z}FG$ which can be viewed as generalizations of the Reynolds ideal $\mathbf{R}FG$. In particular, we are interested in their annihilators. One of our motivations is a result by Tsushima (cf. [16, (59)]) saying that the radical $\mathbf{JZ}FG$ of $\mathbf{Z}FG$ coincides with the annihilator in $\mathbf{Z}FG$ of the sum of all p-elements in G. While the annihilators of our ideals cannot be computed in general, we are able to determine them in a number of specific group algebras, such as group algebras of finite p-groups and group algebras of finite nilpotent groups. We also illustrate the complications that arise in the general case by several examples.

In the last part of the paper, we consider an analogous annihilator problem for blocks instead of group algebras. This leads us to open problems concerning the behaviour of our ideals under perfect isometries and isotypies.

2. Symmetric algebras

Let A be a symmetric algebra over an algebraically closed field F of characteristic $p > 0$, with symmetrizing bilinear form $(.\,|\,.)$. For any F-subspace X of A, we denote its perpendicular subspace by

$$X^\perp := \{y \in A : (x|y) = 0 \quad \text{for} \quad x \in X\}.$$

In [9], we considered the following chain of ideals of the center $\mathbf{Z}A$ of A:

$$\mathbf{Z}A = \mathbf{T}_0A^\perp \supseteq \mathbf{T}_1A^\perp \supseteq \mathbf{T}_2A^\perp \supseteq \ldots \supseteq \mathbf{R}A \supseteq \mathbf{H}A \supseteq \mathbf{Z}_0A \supseteq 0;$$

here $\mathbf{Z}_0A := \sum_B \mathbf{Z}B$ where B ranges over the simple blocks of A. Thus $\mathbf{Z}_0A$ is a direct sum of copies of F, one for each simple block B of A. Moreover, $\mathbf{H}A$ denotes the *Higman ideal* of $\mathbf{Z}A$, i.e. the image of the F-linear map

$$\tau : A \longrightarrow A, \quad x \longmapsto \sum_{i=1}^{n} b_i x a_i,$$

where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are a pair of dual bases of A, i.e. $(a_i|b_j) = \delta_{ij}$ for $i, j = 1, \ldots, n$. Furthermore, $\mathbf{R}A := \mathbf{S}A \cap \mathbf{Z}A$ is the *Reynolds ideal* of $\mathbf{Z}A$; here $\mathbf{S}A$ denotes the socle of A. For every non-negative integer n, $\mathbf{T}_nA^\perp$ is the F-subspace of A perpendicular to

$$\mathbf{T}_nA := \{x \in A : x^{p^n} \in \mathbf{K}A\}$$

where $\mathbf{K}A$ denotes the commutator subspace of A. Thus $\mathbf{K}A$ is spanned by all commutators $ab-ba$ $(a, b \in A)$. For every non-negative integer n, $\mathbf{T}_n A^\perp$ is the image of the semilinear map $\zeta_n : \mathbf{Z}A \longrightarrow \mathbf{Z}A$ defined in the following way: For $z \in \mathbf{Z}A$, $\zeta_n(z)$ is the unique element in $\mathbf{Z}A$ satisfying

$$(\zeta_n(z)|x)^{p^n} = (z|x^{p^n}) \quad \text{for} \quad x \in A.$$

The map ζ_n enjoys the following properties, as shown in [16, (44)-(45)].

Lemma 2.1. *Let m and n be non-negative integers, and let $y, z \in \mathbf{Z}A$. Then the following holds:*
(i) $\zeta_n(y+z) = \zeta_n(y) + \zeta_n(z)$ and $\zeta_n(y)z = \zeta_n(yz^{p^n})$.
(ii) $\zeta_m \circ \zeta_n = \zeta_{m+n}$.

We will make use of the following result [9, Theorem 2.3, Proposition 2.5 and Theorem 4.5].

Theorem 2.2. *(i) $(\mathbf{T}_1 A^\perp)^2 \subseteq \mathbf{H}A$.*
(ii) If p is odd then $(\mathbf{T}_1 A^\perp)^2 = \mathbf{Z}_0 A$.
(iii) If $p = 2$ then $(\mathbf{T}_1 A^\perp)^3 = (\mathbf{T}_1 A^\perp)(\mathbf{T}_2 A^\perp) = \mathbf{Z}_0 A$ and $(\mathbf{T}_1 A^\perp)^2 = \mathbf{Z}A \cdot \zeta_1(1)^2$. Moreover, $\mathbf{Z}B \cdot \zeta_1(1)^2 = F\zeta_1(1)^2 1_B$ for each block B of A.

For the rest of this section, let $p = 2$, and let $e_1, \ldots, e_l$ be representatives for the conjugacy classes of primitive idempotents in A. Then there are uniquely determined elements $r_1, \ldots, r_l \in \mathbf{R}A$ satisfying

$$(e_i|r_j) = \delta_{ij} \quad \text{for} \quad i, j = 1, \ldots, l.$$

These elements $r_1, \ldots, r_l$ are independent of the choice of $e_1, \ldots, e_l$ in their respective conjugacy classes and form an F-basis of $\mathbf{R}A$. In [9, Lemma 3.4], the following result was proved.

Theorem 2.3. *With notation as above, we have*

$$\zeta_1(1)^2 = \sum_{i=1}^{l} (\dim e_i A e_i) \cdot r_i,$$

and $\zeta_1(1)^2 e_i = (\dim e_i A e_i) \cdot e_i r_i$ with $e_i r_i \neq 0$, for $i = 1, \ldots, l$. In particular, $\zeta_1(1)^2 e_i \neq 0$ if and only if $\dim e_i A e_i$ is odd.

The following consequence of Theorem 2.3 will be useful.

Corollary 2.4. *For a block B of A, we have $\zeta_1(1)^2 1_B \neq 0$ if and only if the Cartan matrix of B contains an odd diagonal entry. In particular, we have $\zeta_1(1)^2 \neq 0$ if and only if the Cartan matrix of A contains an odd diagonal entry.*

3. Group algebras

In the following, let G be a finite group, and let (K, R, F) be a splitting p-modular system for G. Thus R is a complete discrete valuation ring, its residue field F is algebraically closed of characteristic $p > 0$, and its field of fractions K has characteristic 0 and contains the $|G|$-th roots of unity. We denote the canonical map $R \longrightarrow F$ by $\alpha \longmapsto \overline{\alpha}$.

The group algebra FG is symmetric; a symmetrizing bilinear form $(.\,|\,.)$ on FG is given by

$$(g|h) := \begin{cases} 1, & \text{if } gh = 1, \\ 0, & \text{otherwise,} \end{cases}$$

for $g, h \in G$. For a subset X of G, we set

$$X^+ := \sum_{x \in X} x \in FG$$

and

$$X^{p^{-n}} := \{y \in G : y^{p^n} \in X\}.$$

An element $g \in G$ is called *real* if g is conjugate to its inverse g^{-1}, and g is said to have *p-defect zero* if $|\mathbf{C}_G(g)|$ is not divisible by p. We denote the set of all real elements of p-defect zero in G by R_G, and we denote the set of conjugacy classes of G by $\mathrm{Cl}(G)$. A conjugacy class of G is called *real* (of *p-defect zero*) if its elements are real (of p-defect zero). In the following, we denote the canonical map $RG \longrightarrow FG$ by $x \longmapsto \overline{x}$.

Proposition 3.1. *Let l, m, n be non-negative integers. Then the following holds:*
(i) $\zeta_n(C^+) = C^{p^{-n}+}$ for $C \in \mathrm{Cl}(G)$; in particular, $\zeta_n(1) = \{1\}^{p^{-n}+}$.
(ii) $\zeta_n(1)1_B = 1_B$ for every simple block B of FG.
(iii) $\zeta_m(1)\zeta_n(1) = \begin{cases} R_G^+, & \text{if } p = 2 \text{ and } m = n = 1, \\ \sum_B 1_B, & \text{otherwise;} \end{cases}$
here B ranges over the simple blocks of FG.
(iv) $\zeta_l(1)\zeta_m(1)\zeta_n(1) = \sum_B 1_B$ where B ranges over the simple blocks of FG.

Proof. (i) is proved in [16, (48)].

(ii) Let C be a conjugacy class of p-singular elements in G. Then $C^+1_B \in \mathbf{Z}B = F1_B$ for every simple block B of FG; in particular, C^+1_B is a linear combination of p-regular elements in G, by Osima's Theorem (cf. [16, (61)]). On the other hand, C^+1_B is a linear combination of elements in G contained in the same p-section as C, by Iizuka's Theorem (cf. [16, (61)]). This implies that $C^+1_B = 0$ for every p-singular conjugacy class C of G. Hence $\zeta_n(1)1_B = \{1\}^{p^{-n}+} \cdot 1_B = 1 \cdot 1_B = 1_B$ by (i).

(iii) The case where $p = 2$ and $m = n = 1$ is handled in [19, Proposition 4.1]. So we suppose that p is odd or that $m > 1$ or that $n > 1$. Then

$$\zeta_m(1)\zeta_n(1) \in (\mathbf{T}_m FG^\perp)(\mathbf{T}_n FG^\perp) = \mathbf{Z}_0 FG$$

by Theorem 2.2. Hence (ii) implies that

$$\zeta_m(1)\zeta_n(1) = \sum_B \zeta_m(1)\zeta_n(1)1_B = \sum_B \zeta_m(1)1_B = \sum_B 1_B$$

where B ranges over the simple blocks of FG.

(iv) By Theorem 2.2, we have $\zeta_l(1)\zeta_m(1)\zeta_n(1) \in (\mathbf{T}_1 FG^\perp)^3 = \mathbf{Z}_0 FG$. The remainder of the proof is similar to that of (iii).

Theorem 2.2 and Proposition 3.1 imply the following.

Corollary 3.2. *(i) If p is odd then $(\mathbf{T}_1 FG^\perp)^2 = \mathbf{Z}_0 FG$.*
(ii) If $p = 2$ then $(\mathbf{T}_1 FG^\perp)^2 = \mathbf{Z}FG \cdot R_G^+$. Moreover, $\mathbf{Z}B \cdot R_G^+ = F \cdot R_G^+ 1_B$ for every block B of FG. In particular, $\dim\, (\mathbf{T}_1 FG^\perp)^2$ equals the number of blocks B of FG such that $R_G^+ \cdot 1_B \neq 0$.

We can combine this with Theorem 2.3 and Corollary 2.4 to obtain:

Theorem 3.3. *Let $p = 2$, let e be a primitive idempotent in FG, and let B be a block of FG. Then the following holds:*
(i) $\dim eFGe$ is odd if and only if $R_G^+ e \neq 0$.
(ii) The Cartan matrix of B contains an odd diagonal entry if and only if $R_G^+ 1_B \neq 0$.

(iii) The Cartan matrix of FG contains an odd diagonal entry if and only if G contains a real element of 2-defect zero.

Note that (i) can also be expressed as follows: Suppose that P is an indecomposable projective left FG-module. Then R_G^+ annihilates P if and only if $\dim \mathrm{End}_{FG}(P)$ is even.

There is no apparent connection between the *number* of real conjugacy classes of 2-defect zero and the *number* of odd diagonal Cartan invariants. For the alternating group A_5 of degree 5 contains three real conjugacy classes of 2-defect zero but only one odd diagonal Cartan invariant (coming from its 2-block of defect zero); indeed, the Cartan matrix of FA_5 for $p = 2$ is

$$\begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(This example will be put into broader perspective in Section 4.) On the other hand, groups of odd order contain precisely one real conjugacy class of 2-defect zero (the trivial one), but all diagonal Cartan invariants in characteristic 2 are odd (even equal to 1).

Let us return to the situation where p is an arbitrary prime. In the following, we choose a set of representatives $g_1, \ldots, g_k$ for the conjugacy classes of G. Then the elements $g_1 + \mathbf{K}FG, \ldots, g_k + \mathbf{K}FG$ form an F-basis of $FG/\mathbf{K}FG$. Moreover, we choose a set of representatives $e_1, \ldots, e_l$ for the conjugacy classes of primitive idempotents in FG. Then $FGe_1, \ldots, FGe_l$ are representatives for the isomorphism classes of indecomposable projective FG-modules. We denote the corresponding principal indecomposable characters by $\Phi_1, \ldots, \Phi_l$ and the corresponding irreducible Brauer characters by $\phi_1, \ldots, \phi_l$.

As in Section 2, there are uniquely determined elements $r_1, \ldots, r_l \in \mathbf{R}FG$ such that $(e_i|r_j) = \delta_{ij}$ for $i, j = 1, \ldots, l$. Moreover, $r_1, \ldots, r_l$ form an F-basis of $\mathbf{R}FG$. For $i = 1, \ldots, l$, the element $e_i + \mathbf{K}FG$ is independent of the choice of e_i in its conjugacy class since $ue_iu^{-1} \equiv u^{-1}ue_i = e_i \pmod{\mathbf{K}FG}$ for every unit u in FG. The elements $e_1, \ldots, e_l$ and $r_1, \ldots, r_l$ can be described in terms of their Brauer characters in the following way.

Lemma 3.4. *For $i = 1, \ldots, l$, the following holds:*
(i) $e_i + \mathbf{K}FG = \sum_{j=1}^{l} \overline{(\Phi_i(g_j^{-1})/|\mathbf{C}_G(g_j)|)}\, g_j + \mathbf{K}FG$.
(ii) $r_i = \sum_{g\in G} \overline{\phi_i(g_{p'}^{-1})}\, g$.

Here $g_{p'}$ denotes the p-factor of g, for $g \in G$.

Proof. (i) Let $\chi : G \longrightarrow K$ be an irreducible character of G. Then the corresponding primitive idempotent e_χ in $\mathbf{Z}KG$ has the form $e_\chi = |G|^{-1} \sum_{g\in G} \chi(1)\chi(g^{-1})g$. Hence $e_\chi \equiv |G|^{-1} \sum_{g\in G} \chi(1)\chi(g^{-1})g \pmod{\mathbf{K}(KG)}$. On the other hand, e_χ is a sum of $\chi(1)$ pairwise orthogonal pairwise conjugate primitive idempotents ϵ_χ in KG. Thus

$$\epsilon_\chi \equiv |G|^{-1} \sum_{g\in G} \chi(g^{-1})g \pmod{\mathbf{K}(KG)},$$

for every primitive idempotent ϵ_χ in KGe_χ. For $i = 1, \ldots, l$, e_i lifts to a primitive idempotent ϵ_i in RG. Moreover, we have

$$\epsilon_i \equiv \sum_{\chi\in \mathrm{Irr}(G)} d_{\chi i}\epsilon_\chi \pmod{\mathbf{K}(KG)}$$

where the integers $d_{\chi i}$ ($\chi \in \mathrm{Irr}(G)$) denote the decomposition numbers. Also, $\mathrm{Irr}(G)$ denotes the set of irreducible characters $\chi : G \longrightarrow K$. Then

$$\epsilon_i \equiv \sum_{g\in G} (\Phi_i(g^{-1})/|G|)g \equiv \sum_{j=1}^{k} (\Phi_i(g_j^{-1})/|\mathbf{C}_G(g_j)|)g_j \pmod{\mathbf{K}(KG)}.$$

Now $\epsilon_i \in RG$ and $\Phi_i(g_j^{-1})/|\mathbf{C}_G(g_j)| \in R$ for $j = 1, \ldots, k$. Since $\mathbf{K}(KG) \cap RG = \mathbf{K}(RG)$ we get

$$\epsilon_i \equiv \sum_{j=1}^{k} (\Phi_i(g_j^{-1})/|\mathbf{C}_G(g_j)|)g_j \pmod{\mathbf{K}(RG)},$$

and (i) follows.

(ii) For $j = 1, \ldots, l$, let $r_j' := \sum_{g \in G} \overline{\phi_j(g_{p'}^{-1})}g$. Then $r_j' \in \mathbf{R}FG$ and

$$(e_i|r_j') = \sum_{h=1}^{k} \overline{(\Phi_i(g_h^{-1})/|\mathbf{C}_G(g_h)|)\phi_j(g_h)} = \overline{|G|^{-1}\sum_{g \in G} \Phi_i(g^{-1})\phi_j(g)} = \delta_{ij}$$

for $i, j = 1, \ldots, l$. Hence $r_j' = r_j$ for $j = 1, \ldots, l$.

Our next result combines Proposition 3.1, Theorem 2.3 and Lemma 3.4.

Proposition 3.5. *Let $p = 2$. Then the following holds:*

$$R_G^+ = \sum_{i=1}^{l} (\dim e_iFGe_i) \cdot r_i.$$

Hence, for $g \in G$, we have

$$\sum_{i=1}^{l} (\dim e_iFGe_i)\overline{\phi_i(g)} = \begin{cases} 1, & \text{if } g \text{ is real of 2-defect zero,} \\ 0, & \text{otherwise.} \end{cases}$$

For later use, the following result will be of interest.

Proposition 3.6. *Let A and B be perfectly isometric 2-blocks of two finite groups G and H, respectively. Then the Cartan matrix of A contains an odd diagonal entry if and only if the Cartan matrix of B does.*

Proof. The proof is similar to the proof of [9, Proposition 3.7]. For the convenience of the reader, we sketch the short argument here. It is known that the Cartan matrices $C = (c_{ij})_{i,j=1}^{l}$ of A and $C' = (c_{ij}')_{i,j=1}^{l}$ of B have the same format, and that they are related by an equation

$$C' = Q \cdot C \cdot Q^\top$$

where $Q = (q_{ij})_{i,j=1}^{l}$ is an integral matrix with determinant ± 1 (cf. [3] or [13]). Thus

$$c_{ii}' = \sum_{j,k=1}^{l} q_{ij}q_{ik}c_{jk} \equiv \sum_{j=1}^{l} q_{ij}^2 c_{jj} \pmod 2$$

for $i = 1, \ldots, l$. If c_{ii}' is odd then c_{jj} has to be odd for some $j \in \{1, \ldots, l\}$ (and conversely).

4. Odd Cartan invariants in even characteristic

In this section we are going to apply the results and methods of Section 3 to specific examples of groups and blocks. We will be particularly interested in odd diagonal Cartan invariants.

In the following, we will always assume that $p = 2$. For $n \geq 3$, the symmetric group $G = S_n$ of degree n contains a real element of 2-defect zero, e.g. an n-cycle (if n is odd) or an $(n-1)$-cycle (if n is even). Hence

Theorem 3.3 implies that the Cartan matrix of S_n in characteristic 2 contains an odd diagonal entry (for $n \geq 3$). (But note that S_n contains a simple 2-block if and only if n is a triangular number, i.e. $n = k(k+1)/2$ for some positive integer k.) We can prove a more specific result by considering 2-blocks of symmetric groups.

Theorem 4.1. *The Cartan matrix C of a 2-block B of a finite symmetric group S_n contains an odd diagonal entry if and only if the weight w of B is even.*

Proof. We will need some basic facts from the representation theory of symmetric groups; most of these can be found in [11]. By a result of Olsson [21], all Cartan invariants (not just the diagonal ones) of a 2-block of odd weight in a symmetric group are even. Thus in the following we concentrate on 2-blocks of even weight.

Enguehard has proved [5] that, for a fixed prime, all blocks of symmetric groups of the same weight are perfectly isometric. Hence it suffices to show, by Proposition 3.6, that for every positive integer w at least one 2-block of weight w in a symmetric group has an odd diagonal Cartan invariant.

Thus we fix a positive integer m and set $w := 2m$, $n := 4m + 1$. Then the principal 2-block B of S_n has weight w and 2-core (1). We will show that the Cartan matrix C of B contains an odd diagonal entry.

The irreducible characters of S_n are labelled by partitions λ of n. For every partition λ of n, we denote the corresponding irreducible character of S_n by χ^λ. Also, we denote by $\chi^\lambda_{\mathrm{reg}}$ the restriction of χ^λ to the set of 2-regular elements in $G = S_n$. We will make use of the fact that the elements $\chi^\lambda_{\mathrm{reg}}$, where λ ranges over the 2-regular partitions with 2-core (1), form a basis for the free abelian group spanned by the irreducible Brauer characters in the principal 2-block of S_n. Thus we get equations

$$\chi^\lambda_{\mathrm{reg}} = \sum_\mu \widetilde{d}_{\lambda\mu} \chi^\mu_{\mathrm{reg}}$$

with integers $\widetilde{d}_{\lambda\mu}$ where μ ranges over the 2-regular partitions of n with 2-core (1). We denote by $\widetilde{D} := (\widetilde{d}_{\lambda\mu})$ the corresponding matrix and set $\widetilde{C} := \widetilde{D}^\top \cdot \widetilde{D}$. Then $\widetilde{D} = DQ$ where D is the usual decomposition matrix of B and Q is a quadratic integral matrix with determinant ± 1. Thus $\widetilde{C} = Q^\top \cdot C \cdot Q$, and it suffices to show (as in the proof of Proposition 3.6) that $\widetilde{C}$ contains an odd diagonal entry. We will in fact show that the entry $\widetilde{c} := \widetilde{c}_{(n)(n)}$ in $\widetilde{C}$ corresponding to the trivial partition (n) of n is odd.

In order to see this, let g be an n-cycle in S_n. It is known that the only partitions λ of n such that $\chi^\lambda(g) \neq 0$ are the hook partitions

$$(n),\ (n-1,1),\ (n-2,2),\ \ldots,(1^n).$$

(This is a special case of the Murnaghan-Nakayama Formula.) All these hook partitions satisfy $\chi^\lambda(g) = \pm 1$. The only 2-regular hook partitions are (n) and $(n-1,1)$, and we observe that $\chi^{(n-1,1)}$ does not belong to B, by the Nakayama Conjecture (as proved by Brauer and Robinson). Note also that the only self-dual hook partition of n is $(2m+1, 1^{2m})$.

By definition, we have

$$\widetilde{c} = \widetilde{c}_{(n)(n)} = \sum_\lambda \widetilde{d}_{\lambda,(n)}$$

where λ ranges over all partitions of n with 2-core (1). Since all 2-regular elements of S_n are contained in the alternating group A_n, $\chi^\lambda_{\mathrm{reg}} = \chi^{\lambda^\top}_{\mathrm{reg}}$ where $\lambda^\top$ denotes the partition of n dual to λ, and therefore $\widetilde{d}_{\lambda,(n)} = \widetilde{d}_{\lambda^\top,(n)}$ for every partition λ of n. Hence $\widetilde{c} = \sum_\lambda \widetilde{d}_{\lambda,(n)}$ where λ ranges over all self-dual partitions of n with 2-core (1).

If $\lambda = (2m+1, 1^{2m})$ then

$$1 \equiv \chi^\lambda(g) \equiv \sum_\mu \widetilde{d}_{\lambda\mu} \chi^\mu(g) \equiv \widetilde{d}_{\lambda,(n)} \pmod 2$$

where μ ranges over the 2-regular partitions of n with 2-core (1); for $\chi^\mu(g) = 0$ if μ is not a hook partition, and $\widetilde{d}_{\lambda\mu} = 0$ if the 2-core of μ is different from (1).

On the other hand, if λ is a self-dual partition of n different from $(2m+1, 1^{2m})$ then, similarly,

$$0 \equiv \chi^\lambda(g) \equiv \sum_\mu \widetilde{d}_{\lambda\mu}\chi^\mu(g) \equiv \widetilde{d}_{\lambda,(n)} \pmod 2.$$

We conclude that $\widetilde{c} \equiv \widetilde{d}_{(2m+1,1^{2m}),(n)} \equiv 1 \pmod 2$, and the result follows.

We can combine Theorem 4.1 with Corollary 3.2 and Theorem 3.3 in order to obtain the following.

Corollary 4.2. *For $G = S_n$ (and $p = 2$), the dimension of $R_G^+ \cdot \mathbf{Z}FG$ equals the number of triangular numbers $m \leq n$ (including $m = 0$) such that $n - m$ is divisible by 4.*

It may be of interest to characterize those partitions λ of n which lead to an odd diagonal Cartan invariant $c_{\lambda\lambda}$. For example, we may ask:

Question 4.3. Suppose that n is divisible by 4, and write $n = 2m$. Is the Cartan invariant $c_{\lambda\lambda}$ corresponding to the 2-regular partition $\lambda = (m+1, m-1)$ of n always odd?

It is known that the dimension of the corresponding simple FS_n-module $D^{(m+1,m-1)}$ is always a power of 2. In fact, $D^{(m+1,m-1)}$ is the reduction mod 2 of the basic spin module of a covering group of S_n.

It is easy to see that the alternating group A_n has an odd diagonal Cartan invariant in characteristic 2, for every $n \geq 5$; the proof is similar to the one for symmetric groups.

In Theorem 4.1, we have produced examples of 2-blocks with odd diagonal Cartan invariants. Now we turn to examples where all diagonal Cartan invariants are even.

Proposition 4.4. *Suppose that B is a 2-block of a finite group G with a non-trivial normal defect group D. Then all diagonal Cartan invariants of B are even.*

Proof. It is known that B is Morita equivalent to a 2-block A of a finite group H containing D as a normal Sylow 2-subgroup (cf. [15]). In particular, A and B have the same Cartan matrix. Thus it suffices to show that all diagonal Cartan invariants of H (in characteristic 2) are even. By Theorem 3.3, it is enough to show that H does not contain real elements of 2-defect zero. So assume that h is a real element of 2-defect zero in H. By Proposition 3.1 (iii), there are elements $t, u \in H$ such that $t^2 = 1 = u^2$ and $tu = h$. Our hypothesis implies that $t \in D$, $u \in D$ and $h = tu \in D$. Since h has 2-defect zero, this is impossible.

Note that Proposition 3.6 and Proposition 4.4 imply that every 2-block A of a finite group which is perfectly isometric to a 2-block with a non-trivial normal defect group has the property that all diagonal Cartan invariants of A are even. In particular, a positive solution to Broué's Abelian Defect Group Conjecture [3] would imply that all diagonal Cartan invariants of 2-blocks with non-trivial *abelian* defect groups are even. We can prove a special case of this without invoking Broué's conjecture.

Proposition 4.5. *Let G be a finite group with elementary abelian Sylow 2-subgroups. Then all odd diagonal Cartan invariants of G (in characteristic 2) correspond to 2-blocks of defect zero (and are therefore equal to 1).*

Proof. Our hypothesis implies that G_2, the set of all 2-elements in G, coincides with $\{1\}^{2^{-1}} = \{g \in G : g^2 = 1\}$. Thus $(G_2^+)^2 = (\{1\}^{2^{-1}+})^2$. But $(G_2^+)^2 = \sum_B 1_B$ where B ranges over all 2-blocks of defect zero in G, by Tsushima's Theorem [16,(83)], and $(\{1\}^{2^{-1}+})^2 = R_G^+$ by Proposition 3.1 (iii). Now let e be a primitive idempotent in FG such that $\dim eFGe$ is odd. Then $0 \neq R_G^+ e$ by Theorem 3.3. Hence $0 \neq 1_B e$ for a 2-block B of defect zero in G, and the result follows.

Important examples of groups with elementary abelian Sylow 2-subgroups are the special linear groups $SL(2, 2^n)$, the smallest Janko group J_1 and the Ree groups $R(q)$. Their Cartan matrices were determined by

Alperin [1], Fong [6] and Landrock-Michler [17, 18]. In the case of $SL(2, 2^n)$, all non-zero Cartan invariants in characteristic 2 are in fact powers of 2, and in the case of J_1 and $R(q)$ the Cartan matrices of the principal 2-blocks are as follows:

$$J_1 : \quad \begin{pmatrix} 8 & 4 & 4 & 4 & 4 \\ 4 & 4 & 1 & 3 & 3 \\ 4 & 1 & 4 & 2 & 2 \\ 4 & 3 & 2 & 4 & 2 \\ 4 & 3 & 2 & 2 & 4 \end{pmatrix} \qquad R(q) : \begin{pmatrix} 4 & 4 & 2 & 2 & 2 \\ 4 & 8 & 3 & 4 & 4 \\ 2 & 3 & 2 & 2 & 2 \\ 2 & 4 & 2 & 4 & 2 \\ 2 & 4 & 2 & 2 & 4 \end{pmatrix}$$

We note that Fong and Harris have proved Broué's Abelian Defect Group Conjecture for principal 2-blocks of finite groups [7], by using the classification of finite simple groups with abelian Sylow 2-subgroups. Hence all diagonal Cartan invariants for these blocks are even, except when the group has odd order.

For an arbitrary prime p and an arbitrary p-block B of positive defect, the number $k(B)$ of ordinary irreducible characters in B, the number $l(B)$ of irreducible Brauer characters in B and the rank of the Cartan matrix of B modulo p are expected to be "locally determined", by variants of Alperin's Weight Conjecture [2, 12]. It seems therefore reasonable to ask the following question.

Question 4.6. Is the existence of an odd diagonal Cartan invariant, for a 2-block of positive defect in a finite group, "locally determined"?

5. Annihilators and group algebras

In this section, let F be an algebraically closed field of characteristic $p > 0$, and let A be a symmetric F-algebra. In Section 2, we looked at the following descending chain of ideals of $\mathbf{Z}A$:

$$\mathbf{Z}A = \mathbf{T}_0 A^\perp \supseteq \mathbf{T}_1 A^\perp \supseteq \mathbf{T}_2 A^\perp \supseteq \ldots \supseteq \mathbf{R}A \supseteq 0.$$

In the following, we will also look at the following ascending chain of ideals of $\mathbf{Z}A$:

$$0 = \mathbf{T}_0\mathbf{Z}A \subseteq \mathbf{T}_1\mathbf{Z}A \subseteq \mathbf{T}_2\mathbf{Z}A \subseteq \ldots \subseteq \mathbf{JZ}A \subseteq \mathbf{Z}A$$

where

$$\mathbf{T}_n\mathbf{Z}A = \{z \in \mathbf{Z}A : z^{p^n} = 0\}$$

for every non-negative integer n. In [9], we showed:

Proposition 5.1. *With notation as above, we have*

$$\mathbf{T}_n\mathbf{Z}A \subseteq \{z \in \mathbf{Z}A : z(\mathbf{T}_n A^\perp) = 0\} \subseteq \{z \in \mathbf{Z}A : z\zeta_n(1) = 0\},$$

for every non-negative integer n.

For all sufficiently large n, we have $\mathbf{T}_n\mathbf{Z}A = \{z \in \mathbf{Z}A : z(\mathbf{T}_n A^\perp) = 0\}$ since $\mathbf{T}_n\mathbf{Z}A = \mathbf{JZ}A$ and $\mathbf{T}_n A^\perp = \mathbf{R}A$ in this case. Also, when $A = FG$ for a finite group G then (for sufficiently large n) Tsushima's Theorem implies that

$$\mathbf{T}_n\mathbf{Z}FG = \{z \in \mathbf{Z}FG : z\zeta_n(1) = 0\}$$

since $\zeta_n(1) = G_p^+$ where G_p denotes the set of p-elements in G (cf. [16, (59)]). For arbitrary n Proposition 5.1 implies the following:

$$\begin{aligned} &\{z \in \mathbf{Z}FG : z^{p^n} = 0\} \\ &\subseteq \{z \in \mathbf{Z}FG : z(C^{p^{-n}+}) = 0 \quad \text{for} \quad C \in \mathrm{Cl}(G)\} \\ &\subseteq \{z \in \mathbf{Z}FG : z\{1\}^{p^{-n}+} = 0\}. \end{aligned}$$

It seems natural to ask when equality holds.

Question 5.2. Are the following conditions equivalent, for $z \in \mathbf{Z}FG$ and every non-negative integer n:

(1) $z^{p^n} = 0$;

(2) $zC^{p^{-n}+} = 0$ for $C \in \mathrm{Cl}(G)$;

(3) $z\{1\}^{p^{-n}+} = 0$?

We first look at examples in characteristic 2.

Example 5.3. Let $p = 2$ and $n = 1$, and let G denote the pullback of

$$S_4 \xrightarrow{\alpha} C_2 \xleftarrow{\beta} C_4$$

where S_4 is the symmetric group of degree 4, $C_4 = \langle c \rangle$ is the cyclic group of order 4 and C_2 is the cyclic group of order 2; moreover, α and β are epimorphisms. Thus G is the subgroup of $S_4 \times C_4$ consisting of all elements xy with $x \in S_4$, $y \in C_4$ and $\alpha(x) = \beta(y)$. Then G is a group of order 48 with $|\mathbf{Z}(G)| = 2$ and $G/\mathbf{Z}(G) \cong S_4$; however, G is not a covering group of S_4 since $\mathbf{Z}(G)$ is not contained in G'.

It is straightforward to verify that $\{1\}^{2^{-1}} = \mathbf{O}_2(G)$, an elementary abelian group of order 8. Let z denote the class sum of the conjugacy class containing the element $(1,2)c$ of order 4. Then $z\mathbf{O}_2(G)^+ = 0$, as is easily computed. Hence (3) holds for z.

On the other hand, let $C = \{c^2\}$, so that $C^{2^{-1}+}$ is the class sum of (1,2) in S_4, multiplied by $c + c^2$. It is easy to check that $zC^{2^{-1}+} = 0$, so (2) does not hold for z.

This example shows that the implication (3) $\Longrightarrow$ (2) does not hold, in general. By Proposition 5.1, the implication (3) $\Longrightarrow$ (1) does not hold in general, either.

Example 5.4. Let $p = 2$ and $n = 1$, and let G denote the pullback of

$$S_4 \xrightarrow{\alpha} C_2 \xleftarrow{\beta} QD_{16}$$

where QD_{16} denotes the quasi-dihedral group of order 16, α and β are epimorphisms, and the kernel of β is a dihedral group of order 8:

$$QD_{16} = \langle a, b : a^8 = 1 = b^2,\ bab^{-1} = a^3 \rangle \quad \text{and} \quad \mathrm{Ker}(\beta) = \langle a^2, b \rangle.$$

Then G is a group of order 192.

Let z be the class sum of the element $(1,2)a$. Then z is the class sum of (1,2) in S_4, multiplied with $a + a^3$. It follows easily that z^2 is the class sum of a conjugacy class consisting of elements of order 12. Thus (1) does not hold for z.

On the other hand, one can check that z annihilates $\mathbf{T}_1 FG^{\perp}$, so that (2) holds for z. This means that the implication (2) $\Longrightarrow$ (1) in Question 5.2 is not true, in general.

It is more difficult to find similar examples in odd characteristic; however, they do exist. Our examples were constructed using the computer algebra system GAP (cf. [8]).

Example 5.5. Let $p = 2$ and $n = 1$, and let H be the semidirect product of an extraspecial group P of order 27 and exponent 3 with $SL(2,3) = Sp(2,3)$. Moreover, let G be the pullback of

$$H \xrightarrow{\alpha} C_3 \xleftarrow{\beta} C_9$$

where C_9 is a cyclic group of order 9, C_3 is a cyclic group of order 3, and α and β are epimorphisms. Then G is a group of order 1944 with 72 conjugacy classes. It is easy to see that $\{1\}^{3^{-1}} = \mathbf{O}_3(G) \cong P \times C_3$. One may check that

$$\dim \mathbf{T}_1\mathbf{Z}FG = 69 = \dim\{z \in \mathbf{Z}FG : z\mathbf{T}_1FG^{\perp} = 0\}$$

(so $\mathbf{T}_1\mathbf{Z}FG = \{z \in \mathbf{Z}FG : z\mathbf{T}_1FG^{\perp} = 0\}$) whereas

$$\dim\{z \in \mathbf{Z}FG : z\{1\}^{3^{-1}+} = 0\} = 71.$$

(Hence $\mathbf{JZ}FG = \{z \in \mathbf{Z}FG : z\{1\}^{3^{-1}+} = 0\}$.) Thus, in this example, the implication (3) $\Longrightarrow$ (2) of Question 5.2 does not hold (but the conditions (1) and (2) are equivalent).

Example 5.6. Let W be a finite reflection group of type E_6. Its rotation subgroup has index 2 in W and is isomorphic to the finite simple group $PSU(4,2)$ of order $25920 = 2^6 \cdot 3^4 \cdot 5$. Thus it coincides with the commutator subgroup W' of W (cf. [10]).

Let L be the root lattice of type E_6. Then L is a free abelian group of rank 6, and $N := L/3L$ is an elementary abelian 3-group of order 3^6. The natural action of W' on N turns N into a uniserial $\mathbf{F}_3W'$-module (where $\mathbf{F}_3$ denotes the field with three elements). One checks that N contains a one-dimensional (trivial) submodule Z, and that N/Z is a simple $\mathbf{F}_3W'$-module of dimension 5.

Let G be the semidirect product of N and W'. One computes that G contains 103 conjugacy classes. Moreover, for $p = 3$, $\mathbf{T}_1\mathbf{Z}FG$ has dimension 101 whereas $\{z \in \mathbf{Z}FG : z(\mathbf{T}_1FG^{\perp}) = 0\}$ has dimension 102. So this example shows that conditions (1) and (2) in Question 5.2 are not equivalent, in general, for an odd prime p.

On the other hand, the conditions in Question 5.2 are equivalent (for $n = 1$) whenever G is a finite symmetric or alternating group, by [20]. We will see below that the conditions (1) and (2) in Question 5.2 are also equivalent whenever G is a finite p-nilpotent group. Next, we rephrase condition (2) in Question 5.2.

Lemma 5.7. *Let $z \in \mathbf{Z}FG$, let $C \in \mathrm{Cl}(G)$, and let n be a non-negative integer. Then $zC^{p^{-n}+} = 0$ if and only if $z^{p^n}C^+$ is an F-linear combination of $G \setminus \{g^{p^n} : g \in G\}$.*

Proof. Let $g \in G$. Then Lemma 2.1 (i) and Proposition 3.1 imply that

$$(g|zC^{p^{-n}+})^{p^n} = (g|z\zeta_n(C^+))^{p^n} = (g|\zeta_n(z^{p^n}C^+))^{p^n} = (g^{p^n}|z^{p^n}C^+).$$

This immediately yields the result.

We can now show that the conditions in Question 5.2 are equivalent whenever G is nilpotent or a suitable Frobenius group.

Proposition 5.8. *Suppose that G is either the direct product of a p-group and a p'-group, or a Frobenius group with kernel K such that p divides $|K|$. Moreover, let n be a non-negative integer and $z \in \mathbf{Z}FG$. Then $z^{p^n} = 0$ if and only if $z\{1\}^{p^{-n}+} = 0$.*

Proof. One direction follows from Proposition 5.1. In order to prove the other, let $z \in \mathbf{Z}FG$ such that $z\{1\}^{p^{-n}+} = 0$. Then, by Lemma 5.7, z^{p^n} is an F-linear combination of $G \setminus \{g^{p^n} : g \in G\}$. On the other hand, [16, (60)] implies that z^{p^n} is an F-linear combination of $\{g^{p^n} : g \in G\}$. Thus $z^{p^n} = 0$.

6. Annihilators and blocks

Let F be an algebraically closed field of characteristic $p > 0$, and let G be a finite group. In this section, we will be concerned with the following question which is a block version of Question 5.2.

Question 6.1. Let B be a block of FG, let $z \in \mathbf{Z}B$, and let n be a non-negative integer. Are the following conditions equivalent:

(1) $z^{p^n} = 0$;

(2) $z(\mathbf{T}_n B^\perp) = 0$?

The examples in Section 5 show that (1) and (2) are not equivalent, in general. In this section, we will prove some positive results. We start with blocks which are nilpotent, in the sense of Broué-Puig [4].

Proposition 6.2. *Let B be a nilpotent block of FG, let $z \in \mathbf{Z}B$, and let n be a non-negative integer. Then $z^{p^n} = 0$ if and only if $z(\mathbf{T}_n B^\perp) = 0$.*

Proof. Let D be a defect group of B. Then $B \cong \mathrm{Mat}(d, FD)$ for a positive integer d, by a Theorem of Puig [22]. By [9, Corollary 5.2], the map

$$f : \mathbf{Z}FD \longrightarrow \mathbf{Z}\mathrm{Mat}(d, FD), \quad z \longmapsto z1_d,$$

is an isomorphism of F-algebras such that $f(\mathbf{T}_n FD^\perp) = \mathbf{T}_n \mathrm{Mat}(d, FD)^\perp$. Let $y \in \mathbf{Z}FD$ and set $z := f(y)$. Then $z^{p^n} = 0$ if and only if $y^{p^n} = 0$. This holds if and only if $y(\mathbf{T}_n FD^\perp) = 0$, by Proposition 5.8. In turn, this is equivalent to $z(\mathbf{T}_n \mathrm{Mat}(d, FD)^\perp) = 0$, and the result follows.

We obtain the following consequence.

Corollary 6.3. *Let G be a finite p-nilpotent group, let $z \in \mathbf{Z}FG$, and let n be a non-negative integer. Then $z^{p^n} = 0$ if and only if $zC^{p^{-n}+} = 0$ for $C \in \mathrm{Cl}(G)$.*

Proof. Since G is p-nilpotent, every block of FG is nilpotent. Thus the result follows from Proposition 6.2 and Proposition 3.1 (i).

In a similar way, we can deal with p-blocks with cyclic defect groups in finite p-solvable groups.

Proposition 6.4. *Let G be a finite p-solvable group, and let B be a block of FG with defect group D. Moreover, let n be a non-negative integer and $z \in \mathbf{Z}B$. Then $z^{p^n} = 0$ if and only if $z(\mathbf{T}_n B^\perp) = 0$.*

Proof. It is well-known (cf. [14], for example) that B is isomorphic to $\mathrm{Mat}(d, FH)$ where d is a positive integer and H is the semidirect product of D and a p'-subgroup E of $\mathrm{Aut}(D)$. As in the proof of Proposition 6.2, the assertion for B is equivalent to the assertion for FH. However, then Proposition 5.8 yields the result since H is a Frobenius group with kernel D.

In a similar way as before, this result has the following consequence.

Corollary 6.5. *Let G be a finite p-solvable group with a cyclic Sylow p-subgroup. Moreover, let n be a non-negative integer and $z \in \mathbf{Z}FG$. Then $z^{p^n} = 0$ if and only if $zC^{p^{-n}+} = 0$ for $C \in \mathrm{Cl}(G)$.*

It would be interesting to know whether the solvability assumption in Proposition 6.4 is really necessary.

Question 6.6. Let G be a finite group, and let B be a block of FG with cyclic defect group. Is $\mathbf{T}_n\mathbf{Z}B = \{z \in \mathbf{Z}B : z(\mathbf{T}_n B^\perp) = 0\}$ for every non-negative integer n?

A positive answer to Question 6.6 would follow from a positive answer to the next question:

Question 6.7. Suppose that A and B are perfectly isometric (or isotypic) blocks of finite groups G and H, respectively. Is there an isomorphism of F-algebras $\mathbf{Z}A \longrightarrow \mathbf{Z}B$ mapping $\mathbf{T}_n A^\perp$ onto $\mathbf{T}_n B^\perp$, for every non-negative integer n?

Some of the background of this question is explained in [3] and [13]. A partial answer to Question 6.7 is provided by the following result which is probably known to the experts but does not seem to appear in the literature.

Proposition 6.8. *Let G and H be finite groups, and let A and B be blocks of FG and FH, respectively, which are perfectly isometric. Then there is an isomorphism of F-algebras $\mathbf{Z}A \longrightarrow \mathbf{Z}B$ mapping $\mathbf{R}A$ onto $\mathbf{R}B$.*

Proof. We denote by $\mathrm{Irr}(A)$ the set of irreducible characters $G \longrightarrow F$ associated with A. Moreover, for $\psi \in \mathrm{Irr}(A)$, we set $r_\psi := \sum_{g\in G} \psi(g^{-1})g$. Then the elements r_ψ, with $\psi \in \mathrm{Irr}(A)$, form an F-basis of $\mathbf{R}A$ (cf. Lemma 3.4).

Suppose that (K, R, F) is a splitting p-modular system for both G and H. The blocks A and B of FG and FH lift to blocks $\widehat{A}$ and $\widehat{B}$ of RG and RH, respectively. We set $K\widehat{A} := K \otimes_R \widehat{A}$ and $K\widehat{B} := K \otimes_R \widehat{B}$.

A perfect isometry between A and B consists of a bijection

$$I : \mathrm{Irr}(K\widehat{A}) \longrightarrow \mathrm{Irr}(K\widehat{B}), \quad \chi \longmapsto I(\chi),$$

together with a map

$$\epsilon : \mathrm{Irr}(K\widehat{A}) \longrightarrow \{\pm 1\}, \quad \chi \longmapsto \epsilon(\chi),$$

satisfying a number of conditions (cf. [3]); here $\mathrm{Irr}(K\widehat{A})$ denotes the set of irreducible characters $G \longrightarrow K$ associated with $K\widehat{A}$. Each $\chi \in \mathrm{Irr}(K\widehat{A})$ defines a primitive idempotent e_χ in $\mathbf{Z}(K\widehat{A})$. Broué [3] has shown that there exists an isomorphism of K-algebras

$$\phi_K : \mathbf{Z}(K\widehat{A}) \longrightarrow \mathbf{Z}(K\widehat{B})$$

sending e_χ to $e_{I(\chi)}$, for $\chi \in \mathrm{Irr}(K\widehat{A})$. Moreover, ϕ_K restricts to an isomorphism of R-algebras $\phi_R : \mathbf{Z}\widehat{A} \longrightarrow \mathbf{Z}\widehat{B}$ and thus induces an isomorphism of F-algebras $\phi_F : \mathbf{Z}A \longrightarrow \mathbf{Z}B$. We set $r_\chi := \sum_{g\in G} \chi(g^{-1})g$ and $c_\chi := |G|/\chi(1)$, so that

$$\phi_K(r_\chi) = (c_\chi/c_{I(\chi)})r_{I(\chi)}$$

for $\chi \in \mathrm{Irr}(K\widehat{A})$. Since a perfect isometry preserves defects of irreducible characters, $c_\chi/c_{I(\chi)}$ is a unit in R, and $\phi_F(\overline{r_\chi}) = \overline{(c_\chi/c_{I(\chi)})r_{I(\chi)}}$ for $\chi \in \mathrm{Irr}(K\widehat{A})$. By the surjectivity of the restriction map, the elements $\overline{r_\chi}$ ($\chi \in \mathrm{Irr}(K\widehat{A})$) span the same F-subspace of $\mathbf{Z}A$ as the elements r_ψ ($\psi \in \mathrm{Irr}(A)$). We conclude that $\phi_F(\mathbf{R}A) = \mathbf{R}B$.

We close with the following special result on finite p-groups.

Proposition 6.9. *Let G be a finite p-group, let $x \in G$, let K be the conjugacy class of x in G, and let K^p be the conjugacy class of x^p in G. Then*

$$(K^+)^p = |\mathbf{C}_G(x^p) : \mathbf{C}_G(x)|(K^p)^+;$$

in particular, $(K^+)^p$ is either zero or a class sum.

Proof. We argue by induction on $|G|$. Suppose first that $x \in \mathbf{Z}(G)$. Then $x^p \in \mathbf{Z}(G)$ and $(K^+)^p = x^p = (K^p)^+$. So we may assume that $x \notin \mathbf{Z}(G)$. Then we choose a maximal subgroup H of G such that $\mathbf{C}_G(x) \subseteq H$. Note that H is normal in G with $|G : H| = p$. Now $\mathbf{C}_G(x) = \mathbf{C}_H(x)$ forces $|G : \mathbf{C}_G(x)| > |H : \mathbf{C}_H(x)|$, so K is a disjoint union of conjugacy classes $K_1, \ldots, K_p$ of H. Let $x_i \in K_i$ for $i = 1, \ldots, p$. Then induction implies that

$$\begin{aligned}(K^+)^p &= (K_1^+ + \ldots + K_p^+)^p = (K_1^+)^p + \cdots + (K_p^+)^p \\ &= \sum_{i=1}^p |\mathbf{C}_H(x_i^p) : \mathbf{C}_H(x_i)|(K_i^p)^+ = |\mathbf{C}_H(x^p) : \mathbf{C}_H(x)| \sum_{i=1}^p (K_i^p)^+.\end{aligned}$$

Suppose first that $\mathbf{C}_G(x^p) = \mathbf{C}_G(x)$. Then also $\mathbf{C}_G(x^p) = \mathbf{C}_H(x^p)$. Thus K^p is the disjoint union of the conjugacy classes $K_1^p, \ldots, K_p^p$ of H. So we obtain

$$(K^+)^p = \sum_{i=1}^{p} (K_i^p)^+ = (K^p)^+ = |\mathbf{C}_G(x^p) : \mathbf{C}_G(x)|(K^p)^+$$

in this case. It remains to deal with the case $\mathbf{C}_G(x^p) > \mathbf{C}_G(x)$. Suppose first that $\mathbf{C}_G(x^p) \neq G$. In this case we may assume $\mathbf{C}_G(x) < \mathbf{C}_G(x^p) \leq H$. We conclude that $\mathbf{C}_H(x) < \mathbf{C}_H(x^p)$ and $(K_i^+)^p = 0$ by induction. Thus also $(K^+)^p = 0 = |\mathbf{C}_G(x^p) : \mathbf{C}_G(x)|(K^p)^+$. So we may assume that $\mathbf{C}_G(x^p) = G$, i.e. $x^p \in \mathbf{Z}(G)$. In this case $K_i^p = \{x^p\}$ for $i = 1, \ldots, p$ and thus $(K^+)^p = 0 = |\mathbf{C}_G(x^p) : \mathbf{C}_G(x)|(K^p)^+$.

Acknowledgments. The ideas in this paper have their origin in visits of B. Külshammer to the National University of Ireland, Maynooth, and to the Technical University of Budapest, in September 2003. B. Külshammer is very grateful for the invitation to Maynooth and for the hospitality received there. His visit to Maynooth was partially funded by a New Researcher Award from the National University of Ireland, Maynooth. B. Külshammer's visit to Budapest was kindly supported by the German-Hungarian exchange project No. D-4/99 (TéT-BMBF) and by the Hungarian National Science Foundation Research Grants T034878 and T042481.

References

1. J.L. Alperin, Projective modules for $SL(2, 2^n)$, *J. Pure Appl. Algebra* **15** (1979), 219-234
2. J.L. Alperin, Weights for finite groups, *Proc. Symp. Pure Math.* **47** (1987), Part 1, 369-379
3. M. Broué, Isométries parfaites, types de blocs, catégories dérivée, *Astérisque* **181-182** (1990), 61-92
4. M. Broué and L. Puig, A Frobenius theorem for blocks, *Invent. Math.* **56** (1980), 117-128
5. M. Enguehard, Isométries parfaites entre blocs de groupes symétriques, *Astérisque* **181-182** (1990), 157-171
6. P. Fong, On decomposition numbers of J_1 and $R(q)$, Symposia Mathematica Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome 1972), pp. 415-422, Academic Press, London 1974
7. P. Fong and M.E. Harris, On perfect isometries and isotypies in finite groups, *Invent. Math.* **114** (1993), 139-191
8. The GAP group, Aachen, St. Andrews, "GAP - Groups, Algorithms, and Programming," Version 4.3, 2002 [`http://www-gap.dcs.st-and.ac.uk/~gap`]
9. L. Héthelyi, E. Horváth, B. Külshammer and J. Murray, Central ideals and Cartan invariants of symmetric algebras, preprint
10. J.E. Humphreys, Reflection groups and Coxeter groups, Cambridge University Press, Cambridge 1990
11. G. James and A. Kerber, The representation theory of the symmetric group, Addison-Wesley, London 1981
12. R. Knörr and G.R. Robinson, Some remarks on a conjecture of Alperin, *J. London Math. Soc.* (2) **39** (1989), 48-60
13. S. König and A. Zimmermann, Derived equivalences for group rings, Springer-Verlag, Berlin 1998
14. B. Külshammer, On p-blocks of p-solvable groups, *Comm. Algebra* **9** (1981), 1763-1785
15. B. Külshammer, Crossed products and blocks with normal defect groups, *Comm. Algebra* **13** (1985), 147-168
16. B. Külshammer, Group-theoretical descriptions of ring-theoretical invariants of group algebras, *Progress in Math.* **95** (1991), 425-442
17. P. Landrock and G.O. Michler, Block structure of the smallest Janko group, *Math. Ann.* **232** (1978), 205-238

18. P. Landrock and G.O. Michler, Principal 2-blocks of the simple groups of Ree type, *Trans. Amer. Math. Soc.* **260** (1980), 83-111

19. J.C. Murray, Blocks of defect zero and products of elements of order p, *J. Algebra* **214** (1999), 385-399

20. J. Murray, Squares in the centre of the group algebra of a symmetric group, *Bull. London Math. Soc.* **34** (2002), 155-164

21. J.B. Olsson, Lower defect groups in symmetric groups, *J. Algebra* **104** (1986), 37-56

22. L. Puig, Nilpotent blocks and their source algebras, *Invent. Math.* **93** (1988), 77-116